Exercises on Propositional Logic

Deduction

Exercise Three weird implications

$\neg A \rightarrow B$
$B \rightarrow A$
$A \rightarrow (C \wedge D)$

4. Prove the proposition $A \wedge C \wedge D$ using Modus Ponens only, or explain why this is not possible.

Exercise Three weird implications

$\neg A \rightarrow B$
$B \rightarrow A$
$A \rightarrow (C \wedge D)$

4. Prove the proposition $A \wedge C \wedge D$ using Modus Ponens only, or explain why this is not possible.

It is not possible to prove $A \wedge C \wedge D$ using Modus Ponens only. This follows from the fact that Modus Ponens is not applicable to any pair of formulas in the knowledge base.

Exercise Three weird implications

$\neg A \rightarrow B$
$B \rightarrow A$
$A \rightarrow (C \wedge D)$

5. Prove the proposition $A \wedge C \wedge D$ using resolution.

Exercise Three weird implications

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5. Prove the proposition $A \wedge C \wedge D$ using resolution.

Proof by refutation:

1. $A \vee B$ premise
2. $\neg B \vee A$ premise
3. $\neg A \vee C$ premise
4. $\neg A \vee D$ premise
5. $\neg A \vee \neg C \vee \neg D$ negated thesis
6. A resolution 1, 2
7. C resolution 3, 6
8. D resolution 4, 6
9. $\neg C \vee \neg D$ resolution 5, 6
10. $\neg D$ resolution 7, 9
11. $[]$ resolution 8, 10

Exercise It rains, again

Consider the following set of propositional formulas
1.$Rains \rightarrow Wet$
2.$Wet \rightarrow \neg Rains$
3 $Rains$

(a) Prove, via tableaux and via resolution, that the given set is inconsistent.

(b) Prove, via tableaux and via resolution, that all the pairs of formulas, are consistent.

Exercise It rains, again - Resolution

1.$Rains \rightarrow Wet$
2.$Wet \rightarrow \neg Rains$
3 $Rains$

With resolution:
1.$\neg Rains \vee Wet$
2. $\neg Wet \vee \neg Rains$
3. $Rains$
(a) The given set is inconsistent:
4. $\neg Rains$ from 1 and 2
5. $\{\}$ from 3 and 4

(b) From 1 and 2 we get $\neg Rains$; from 1 and 3 we get $\neg Wet$; from 2 and 3 we get $\neg Wet$, so no pair is inconsistent.

Exercise It rains, again - Tableau 1

1.$Rains \rightarrow Wet$
2.$Wet \rightarrow \neg Rains$
3 $Rains$

$$
\begin{array}{c}
R \rightarrow W \\
\hline
W \rightarrow \neg R \\
\hline
R \\
\hline
\begin{array}{cc}
\neg R \star & \begin{array}{c} W \\ \hline \begin{array}{cc} \neg W \star & \neg R \star \end{array} \end{array}
\end{array}
\end{array}
$$

The tableau closes, hence the three formulas are inconsistent.

Exercise It rains, again - Tableau 2

1.$Rains \to Wet$
2.$Wet \to \neg Rains$

$$\boxed{\dfrac{\dfrac{R \to W}{W \to \neg R}}{\dfrac{\neg R}{\neg W \qquad \neg R} \qquad \dfrac{W}{\neg W \star \qquad \neg R}}}$$

The tableau is open, hence the two formulas are consistent.

Exercise It rains, again - Tableau 3

1.$Rains \rightarrow Wet$
3 $Rains$

$$\begin{array}{c} R \rightarrow W \\ \hline R \\ \hline \neg R \star \qquad W \end{array}$$

The tableau is open, hence the two formulas are consistent.

Exercise It rains, again - Tableau 4

2.$Wet \rightarrow \neg Rains$
3 $Rains$

With tableaux

$$\begin{array}{c} W \rightarrow \neg R \\ \hline R \\ \hline \neg W \qquad \neg R \star \end{array}$$

The tableau is open, hence the two formulas are consistent.

Exercise Two tableaux

Let A, B, C be propositional symbols. Verify, using tableaux, whether or not:

1) $(A \rightarrow C) \wedge (B \rightarrow C)$ entails $(A \vee B) \rightarrow C$

2) $(A \vee B) \rightarrow C$ entails $(A \rightarrow C) \wedge (B \rightarrow C)$

Exercise Two tableaux

Let A, B, C be propositional symbols. Verify, using tableaux, whether or not:

1) $(A \to C) \wedge (B \to C)$ entails $(A \vee B) \to C$

Question 1: the answer is positive.

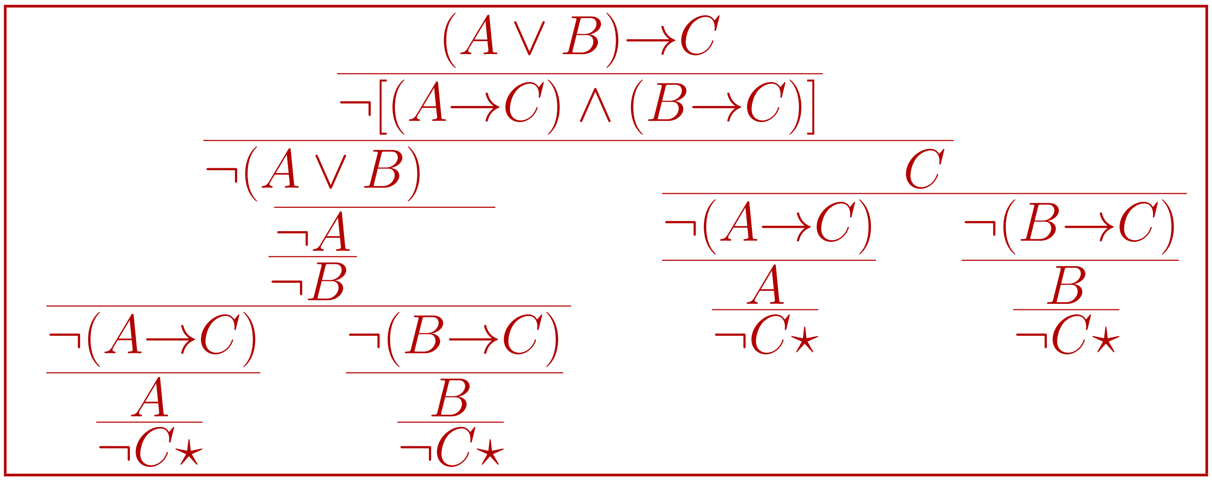
$$
\begin{array}{c}
(A \to C) \wedge (B \to C) \\ \hline
\neg[(A \vee B) \to C] \\ \hline
(A \vee B) \\ \hline
\neg C \\ \hline
A \to C \\ \hline
B \to C \\ \hline
\begin{array}{cc}
\begin{array}{c}
\neg A \\ \hline
\begin{array}{cc}
\begin{array}{c}
\neg B \\ \hline
\begin{array}{cc} A\star & B\star \end{array}
\end{array} & C\star
\end{array}
\end{array} & C\star
\end{array}
\end{array}
$$

Exercise Two tableaux

Let A, B, C be propositional symbols. Verify, using tableaux, whether or not:

2) $(A \vee B) \to C$ entails $(A \to C) \wedge (B \to C)$

Question 2: the answer is positive.



Exercise The Climber

Given the following symbols and sentences:
C to indicate that Gianni is a climber;
F to indicate that Gianni is fit;
L to indicate that Gianni is lucky:
E to indicate that Gianni climbs mount Everest.

If Gianni is a climber and he is fit, he climbs mount Everest.
If Gianni is not lucky and he is not fit, he does not climb mount Everest.
Gianni is fit.

(a) Formalize the above sentences in propositional logic.

Exercise The Climber

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C to indicate that Gianni is a climber;
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If Gianni is a climber and he is fit, he climbs mount Everest.
If Gianni is not lucky and he is not fit, he does not climb mount Everest.
Gianni is fit.

(a)Formalize the above sentences in propositional logic.

$(C \wedge F) \rightarrow E$
$(\neg L \wedge \neg F) \rightarrow \neg E$
F

Exercise The Climber

$(C \wedge F) \rightarrow E$
$(\neg L \wedge \neg F) \rightarrow \neg E$
F

(b) Tell if the KB buit in (a) is consistent, and tell if some of the following sets are models for the above sentences:
$\{\ \}$; $\{C, L\}$; $\{L, E\}$; $\{F, C, E\}$; $\{L, F, E\}$.

Exercise The Climber

$(C \wedge F) \rightarrow E$
$(\neg L \wedge \neg F) \rightarrow \neg E$
F

(b) Tell if the KB buit in (a) is consistent, and tell if some of the following sets are models for the above sentences:
$\{\ \}$; $\{C, L\}$; $\{L, E\}$; $\{F, C, E\}$; $\{L, F, E\}$.

(b) The KB is consistent: it has at least a model, as the following check shows.

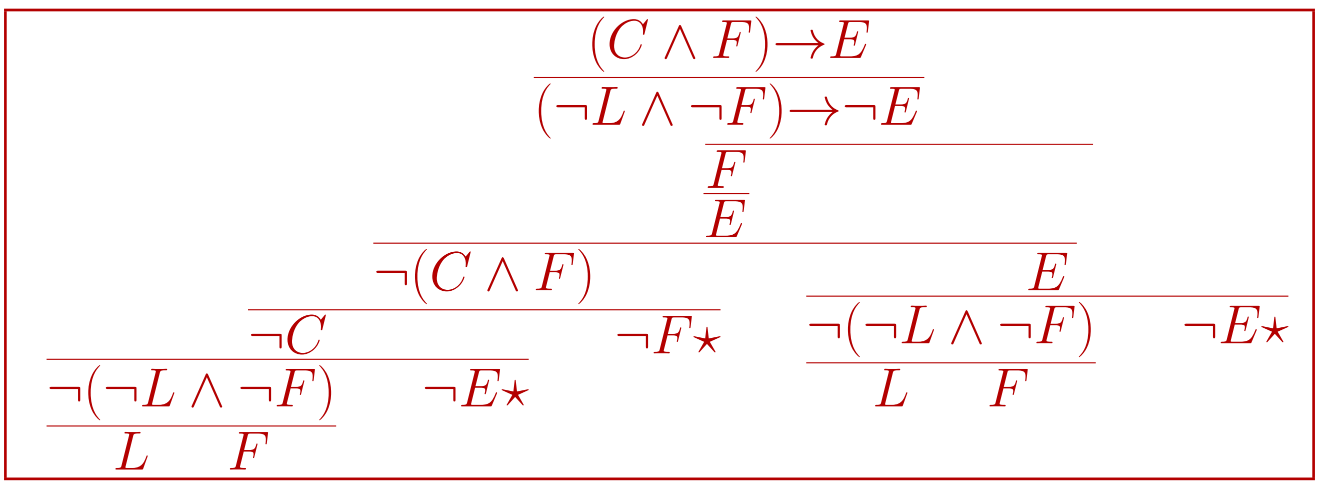
1. $\{\ \}$ is not a model (it models 1 and 2 but not 3);
2. $\{C, L\}$ is not a model (it models 1 and 2 but not 3);
3. $\{L, E\}$ is not a model (it models 1 and 2 but not 3);
4. $\{F, C, E\}$ is a model;
5. $\{L, F, E\}$ is a model.

Exercise The Climber

$(C \wedge F) \rightarrow E$
$(\neg L \wedge \neg F) \rightarrow \neg E$
F

(c) Verify, using tableaux, if it is consistent with the KB built in (a) that Gianni climbs mount Everest.

(c) It is consistent (see figure, the tableau is complete and open);



Exercise The Climber

$(C \wedge F) \rightarrow E$
$(\neg L \wedge \neg F) \rightarrow \neg E$
F

(d) Verify, using tableaux, if it is derivable from the KB built in (a) that Gianni climbs mount Everest. In the negative case, give a formula that added to the KB built in (a) gives a KB' that is still consistent and makes the derivation possible.

Exercise The Climber

$(C \wedge F) \rightarrow E$
$(\neg L \wedge \neg F) \rightarrow \neg E$
F

(d) It is not derivable (see figure) it becomes derivable if we add C. In fact adding C preserves the consistency of the KB and makes the tableau to close.

$$(C \wedge F) \rightarrow E$$
$$(\neg L \wedge \neg F) \rightarrow \neg E$$
$$F$$
$$\neg E$$
$$\neg(C \wedge F) \qquad E\star$$
$$\neg C \qquad \neg F\star$$
$$\neg(\neg L \wedge \neg F) \qquad \neg E$$
$$L \qquad F$$

Exercise A very long formula

Using tableaux, state if the following formula is valid, contradictory or satisfiable (in the last case show a model).

$$\big((P \vee Q) \wedge (Q \rightarrow R) \wedge (R \rightarrow \neg P)\big) \rightarrow (P \leftrightarrow \neg Q)$$

Exercise A very long formula

Using tableaux, state if the following formula is valid, contradictory or satisfiable (in the last case show a model).

$$((P \vee Q) \wedge (Q \rightarrow R) \wedge (R \rightarrow \neg P)) \rightarrow (P \leftrightarrow \neg Q)$$

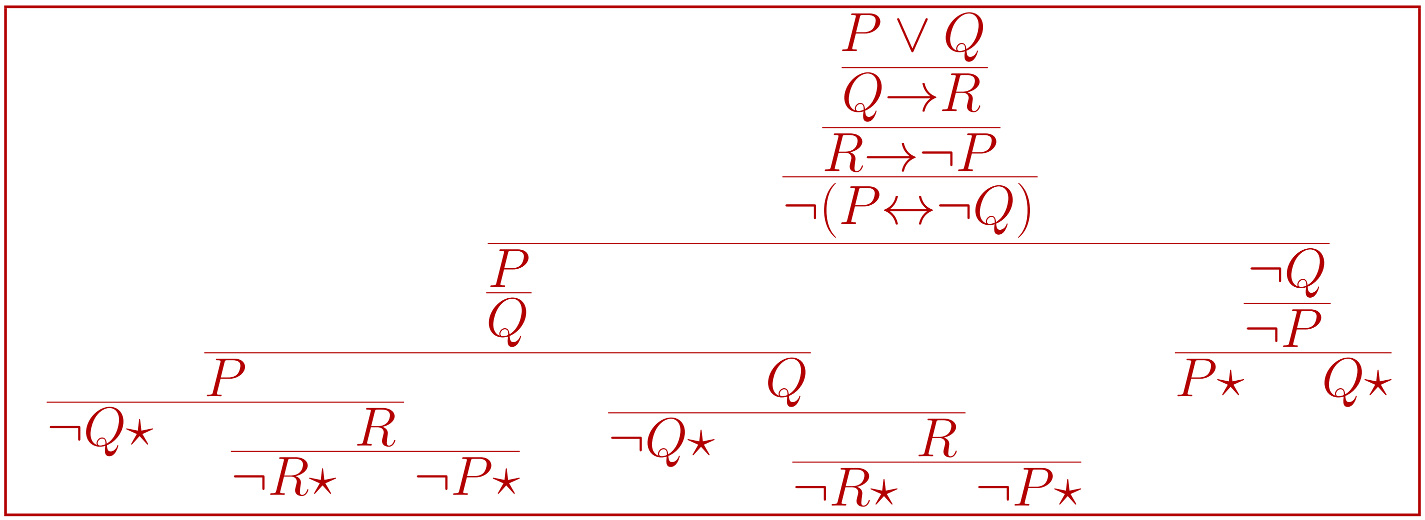
Try validity first, as if it is valid, I have also proved that it is satisfiable and not contradictory.

Exercise A very long formula

Using tableaux, state if the following formula is valid, contradictory or satisfiable (in the last case show a model).

$$((P \vee Q) \wedge (Q \rightarrow R) \wedge (R \rightarrow \neg P)) \rightarrow (P \leftrightarrow \neg Q)$$

Negate the formula and build the tableau.



The formula is valid, as the tableau for its negation closes.

Exercise

Tell whether the propositional formula

$$[(A \to C) \vee (B \to C)] \to [(A \wedge B) \to C]$$

a) is satisfiable
b) is valid
c) is a contraddiction.

Answer a), b, and c) using the tableau method.

Try validity first, as if it is valid, I have also proved that it is satisfiable and not contradictory.

Exercise

So negate the formula: $\neg\{[(A\to C)\vee(B\to C)]\to[(A\wedge B)\to C]\}$

$$
\begin{array}{c}
[(A\to C)\vee(B\to C)] \\
\hline
\neg[(A\wedge B)\to C] \\
\hline
A\wedge B \\
\hline
\neg C \\
\hline
A \\
\hline
B \\
\hline
\begin{array}{cc}
\begin{array}{c} A\to C \\ \hline \neg A\star \quad C\star \end{array} & \begin{array}{c} B\to C \\ \hline \neg B\star \quad C\star \end{array}
\end{array}
\end{array}
$$

The formula is valid, as the tableau for its negation closes.

The tableau for the negated formula closes, hence it is valid, which implies that the formula is satisfiable, and not a contradiction.

Exercise

Prove, using propositional tableaux, that
$\{A \to (B \vee C), \neg B\}$ entails $A \to C$.

Negate the thesis and build the tableau.

$$
\begin{array}{c}
A \to (B \vee C) \\
\neg B \\
\neg(A \to C) \\
A \\
\neg C \\
\hline
\begin{array}{cc}
\neg A \star & \begin{array}{c} B \vee C \\ \hline \begin{array}{cc} B \star & C \star \end{array} \end{array}
\end{array}
\end{array}
$$

The tableau closes, hence the entailment is proved.

Exercise

Let A, B, C be propositional symbols. Given $KB = \{A \rightarrow C, \;\; B \rightarrow C, \;\; A \vee B\}$, tell wether C can be derived from KB or not in each one of the following cases

a. with Modus Ponens;
b. with Resolution.

In each case, if the answer is positive, show the derivation; if it is negative motivate it.

Solution

Let A, B, C be propositional symbols. Given $KB = \{A \rightarrow C, \;\; B \rightarrow C, \;\; A \vee B\}$, tell wether C can be derived from KB or not in each one of the following cases

a. with Modus Ponens;

a. C cannot be derived using MP as we know that $A \vee B$, but we do not know wether A or B is true, so we cannot apply MP to A and $A \rightarrow C$, and we cannot apply MP to B and $B \rightarrow C$.

Solution

Let A, B, C be propositional symbols. Given $KB = \{A \rightarrow C, \;\; B \rightarrow C, \;\; A \vee B\}$, tell wether C can be derived from KB or not in each one of the following cases

b. with Resolution.

b. C can be derived with Resolution
1. $\neg A \vee C$
2. $\neg B \vee C$
3. $A \vee B$
4. $\neg C$ negated thesis
5. $B \vee C$ from 1. and 3.
6. C from 2. and 5.
7. $\{\ \}$ from 4. and 6.

Exercise

Tell whether the following propositional formulas are (a) satisfiable, (b) valid, (c) contraddictiory.
In case (a) show a model. Use the tableau method.

HINT: Try validity first, as if the formula is valid, we have also proved that it is satisfiable and not contradictory.

1. $(P \wedge R) \leftrightarrow (R \wedge P)$
2. $(P \vee \neg P) \leftrightarrow (P \wedge \neg P)$
3. $\neg P \rightarrow (P \wedge \neg P)$
4. $(P \vee Q) \leftrightarrow P$

Solution 1

$$
\begin{array}{c}
\neg[(P \wedge R) \leftrightarrow (R \wedge P)] \\
\hline
\begin{array}{cc}
\begin{array}{c}
P \wedge R \\
\neg(R \wedge P) \\
\hline
P \\
R \\
\hline
\begin{array}{cc} \neg R \star & \neg P \star \end{array}
\end{array}
&
\begin{array}{c}
\neg(P \wedge R) \\
R \wedge P \\
\hline
R \\
P \\
\hline
\begin{array}{cc} \neg R \star & \neg P \star \end{array}
\end{array}
\end{array}
\end{array}
$$

The tableau is closed.

$(P \wedge R) \leftrightarrow (R \wedge P)$ is valid.

Solution 2.1

$$
\dfrac{\neg[(P \vee \neg P)\leftrightarrow(P \wedge \neg P)]}{\dfrac{\dfrac{P \vee \neg P}{\neg(P \wedge \neg P)}}{\dfrac{P}{P \qquad \neg P\star} \qquad \dfrac{\neg P}{P \qquad \neg P\star}} \qquad \dfrac{\dfrac{\dfrac{\neg(P \vee \neg P)}{P \wedge \neg P}}{\neg P}}{P\star}}
$$

The tableau is open.

$(P \vee \neg P)\leftrightarrow(P \wedge \neg P)$ is not valid.

Solution 2.2

$$\cfrac{(P \vee \neg P) \leftrightarrow (P \wedge \neg P)}{\begin{array}{c|c} \cfrac{\cfrac{\cfrac{P \vee \neg P}{P \wedge \neg P}}{P}}{\neg P \star} & \cfrac{\cfrac{\cfrac{\neg(P \vee \neg P)}{\neg(P \wedge \neg P)}}{\neg P}}{P \star} \end{array}}$$

The tableau is closed.

$(P \vee \neg P) \leftrightarrow (P \wedge \neg P)$ is contraditory.

Solution 3.1

$$\boxed{\begin{array}{c} \neg[\neg P \rightarrow (P \wedge \neg P)] \\ \hline (P \vee Q) \\ \hline P \\ \hline \neg(P \wedge \neg P) \\ \hline \neg P\star \qquad P \end{array}}$$

The tableau is open.

$\neg P \rightarrow (P \wedge \neg P)$ is not valid.

Solution 3.2

$$\begin{array}{ll} \multicolumn{2}{l}{\neg P \rightarrow (P \wedge \neg P)} \\ \hline P & \begin{array}{c} (P \wedge \neg P) \\ \hline P \\ \hline \neg P \star \end{array} \end{array}$$

The tableau is open.

$\neg P \rightarrow (P \wedge \neg P)$ is satisfied by { P}

Solution 4.1

$$\boxed{\begin{array}{c}
\neg[(P \vee Q) \leftrightarrow P] \\
\hline
\begin{array}{cc}
\begin{array}{c}
(P \vee Q) \\
\hline
P \\
\hline
\neg P \\
\hline
P\star \quad\quad Q
\end{array}
&
\begin{array}{c}
\neg(P \vee Q) \\
\hline
P \\
\hline
\neg P \\
\hline
Q\star
\end{array}
\end{array}
\end{array}}$$

The tableau is open.

$(P \vee Q) \leftrightarrow P$ is not valid.

Solution 4.2

$$
\begin{array}{cc}
\multicolumn{2}{c}{(P \vee Q) \leftrightarrow P} \\
\hline
\begin{array}{c}
(P \vee Q) \\
\hline
P \\
\hline
\begin{array}{cc} P & \quad Q \end{array}
\end{array}
&
\begin{array}{c}
\neg(P \vee Q) \\
\hline
\neg P \\
\hline
\neg P \\
\hline
\neg Q
\end{array}
\end{array}
$$

The tableau is open.

$(P \vee Q) \leftrightarrow P$ is satisfied by $\{$ P$\}$, $\{$ P,Q$\}$, $\{$ $\}$.

Exercise

Prove, using propositional tableaux, that
$\{A \to (B \vee C), \neg B\}$ entails $A \to C$.

Exercise

Prove, using propositional tableaux, that
$\{A \to (B \vee C), \neg B\}$ entails $A \to C$.

Negate the thesis $\neg(A \to C)$.

$$
\begin{array}{c}
A \to (B \vee C) \\
\neg B \\
\neg(A \to C) \\
A \\
\neg C \\
\hline
\begin{array}{cc}
\neg A \star & \begin{array}{c} B \vee C \\ \hline \begin{array}{cc} B \star & C \star \end{array} \end{array}
\end{array}
\end{array}
$$

The tableau is closed.