Implementation of Saturating Theorem Provers

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Proofs

\[ h(s) \land \neg m(s) \]

Logic Stuff

\[ \forall X (\neg h(X) \lor m(X)) \]

Fig. 1.

International Summer School on Satisfiability, Satisfiability Modulo Theories, and Automated Reasoning
Abstract

We will describe several aspects of the implementation of a saturating, superposition-based theorem prover for first-order logic. We discuss the basic architecture of a prover, and the organization of the actual proof search via the given-clause algorithm. Simplification and redundancy elimination are, in practice, critical, and we describe how these can be integrated into the basic proof procedure.

Another topic is that of terms and clauses, the most basic data objects in any prover, and how they can be implemented. We discuss bottlenecks of naive implementations, as well as the principle and some examples of indexing techniques to overcome these bottlenecks. The presentation concludes with a look at the influence of search heuristics.
Introduction
Black Box View

Axioms → Clausification → CNF → Saturation → Proof Extraction → Proof (?)

Conjecture → Clausification → CNF → Saturation → Proof Extraction → Proof (?)
Set of clauses $S$
(including conjecture clauses)

Saturation

Empty clause (?)
Blue Box View

Set of clauses $S$
(including conjecture clauses)

Saturation

- Basic datatypes (terms, clauses,...)
- Saturation algorithm
- Term- and clause indexing
- Search control

Empty clause (?)
Proving by Saturation

- **Goal:** Show *unsatisfiability* of a set of clauses $S$
- **Approach:**
  - Systematically enrich $S$ with clauses derived via *inferences* from clauses in $S$ (*Saturation*)
  - Optionally: Remove/simplify *redundant* clauses
- **Outcome:**
  - Derivation of the empty clause $\square$ (explicit witness of unsatisfiability)
  - Successful saturation (up to redundancy): $S$ is satisfiable
  - ... or infinite sequence of derivations
- **Properties:**
  - Correctness: Only logical consequences are derived
  - Completeness: Every unsatisfiable $S$ will eventually lead to the derivation of $\square$
(Equality Resolution)
\[
\frac{u \not\equiv v \lor R}{\sigma(R)} \quad \text{if } \sigma = \text{mgu}(u, v),
\]
\[
\ldots
\]

(Superposition into negative literals)
\[
\frac{s \simeq t \lor S \quad u \not\equiv v \lor R}{\sigma(u[p \leftarrow t] \not\equiv v \lor S \lor R)} \quad \text{if } \sigma = \text{mgu}(u|_p, s), \quad \sigma(s) \not\equiv \sigma(t), \ldots
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(Rewriting of negative literals)
\[
\frac{s \simeq t \quad u \not\equiv v \lor R}{s \not\equiv s \lor R} \quad \text{if } u|_p = \sigma(s) \text{ and } \sigma(s) > \sigma(t)
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\frac{\sigma(s) \not\equiv \sigma(t)}{\ldots}
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Generating inference rules

- Necessary for completeness
- Increase size of proof state

Simplification rules

- Critical for performance
- Reduce size of proof state

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\vdots
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\[\frac{s \not\equiv R}{R}\]

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- Easy to keep track of
- Cheap to implement

**Non-local (multiple premises)**
- Harder to keep track of (pairs of clauses!)
- Expensive to implement (find partners)

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>99% of generated clauses

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\]

>90% of cpu time
Basic Data Types
Running Example

- **Clauses:**
  - $C_1 = X \not\approx 0 \lor add(X, s(Y)) \approx s(Y)$
  - $C_2 = add(s(X), Y) \approx s(add(X, Y))$

- **Properties:**
  - $C_1$ is a non-unit Horn clause (at most one positive literal)
  - $C_2$ is a positive unit-clause (one literal) literal
  - Both are purely equational
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- **Basic components:**
  - *add* is a *function symbol* of arity 2
  - *s* is a function symbol of arity 1
  - *0* is a constant (function symbol of arity 0)
  - *X, Y* are variables (implicitly universally quantified)
  - $\approx$ represents the equality relation
  - $\not\approx$ represents the negated equality relation
  - $\lor$ is the disjunctive operator (logical *or*)
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- **Terms and literals:**
  - $X, Y, 0$ are elementary terms
  - $s(X)$ is a (composite) term
  - So are $add(s(X), Y), s(add(X, Y)), add(X, s(Y)), \ldots$
  - $add(s(X), Y) \equiv s(add(X, Y))$ is a positive literal
  - $add(X, s(Y)) \equiv s(Y)$ is a positive literal
  - $X \not\equiv 0$ is a negative literal
Function symbols (and predicate symbols, if used) are usually represented by small positive integers!

<table>
<thead>
<tr>
<th>Enc.</th>
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<th>Arity</th>
<th>Remarks</th>
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<tbody>
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**Signature table**

- Associates function symbol with index (small integer)
- Can store additional information
- Implement as array: Fast look-up (O(1)) by index
- Add e.g. tree for fast mapping $name \rightarrow index$
typedef struct funccell
{
    /* f_code is implicit by position in the array */
    char* name;
    int arity;
    ...
    Type_p type;    /* Simple type of the symbol */
    FunctionProperties properties;
} FuncCell, *Func_p;

typedef struct sigcell
{
    long size;    /* Size of the array */
    FunCode f_count;    /* Largest used f_code */
    Func_p f_info;    /* The array */
    StrTree_p f_index;    /* Back-assoc: symbol⇒index */
    ...
    TypeTable_p type_table;
    ...
} SigCell, *Sig_p;
Function symbols (and predicate symbols, if used) are usually represented by small positive integers!
Variables

Function symbols (and predicate symbols, if used) are usually represented by small **positive** integers!

- Variables have no persistent names
  - Each clause is individually universally quantified
  - Scope of a variable name is one clause
- Frequent encoding: Small **negative** integers
  - $-1$ is the first variable in a clause
  - $-3$ is the second variable in a clause
  - $-5$ is the third variable in a clause
  - $\ldots$
  - Temporary association $\text{index} \leftrightarrow \text{name}$ for parsing
Variables

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  - ... Temporary association $\text{index} \leftrightarrow \text{name}$ for parsing
- This leaves even numbers for alternative variables
  - When two clauses interact via unification, the usually need disjoint variable sets!
Terms are Trees

- Terms are ordered trees
  - Leaves are labeled constants or variables
  - Internal nodes are labeled with non-constant symbols
  - Node with symbol of arity \( n \) has \( n \) children

\[
\text{add}(s(Y), X) \Rightarrow \text{add} \quad s \quad X
\]

\[
\text{add} \quad s \quad X
\]

\[
\text{add} \quad s \quad X
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Enc.
---
-1  X
-3  Y
-5  Z

\[
\text{add}(s(Y), X) \implies \begin{array}{c}
\text{add} \\
\downarrow \\
\text{s} \\
\downarrow \\
Y \\
\end{array} \quad \begin{array}{c}
\downarrow \\
X \\
\downarrow \\
\text{3} \\
\downarrow \\
-3 \\
\end{array} \quad \begin{array}{c}
\downarrow \\
\text{2} \\
\downarrow \\
\text{-1} \\
\end{array}
\]
Term Representations

- Cool languages have s-expressions: Terms for free
  - LISP/Scheme: \( \text{add}(s(Y),X) \rightarrow (\text{add}(s\ Y)\ X) \)
  - Python: \( \text{add}(s(Y),X) \rightarrow [2\ [3\ -3]\ -1] \)

- Cooler languages have recursive data types
  - ML/Caml/OCaml
  - Haskell
Term Representations

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- C has pointers

```
typedef struct termcell {
    FunCode       f_code;         /* Top symbol of term */
    TermProperties properties;   /* Like basic, lhs, top */
    int           arity;         /* Redundant, but saves around the signature time */

    struct termcell** args;     /* Pointer to array of arguments */
    ...
} TermCell, *Term_p, **TermRef;
```
Meta-information plus two terms plus next pointer

This reflects the pure equational paradigm of E

Alternative: Atoms are terms (with a predicate symbol as the top symbol)

- Literals have only one term pointer
- Equality is just a special predicate symbol

```c
typedef struct eqnCell {
    EqnProperties properties; /* Positive, maximal, eq. */
    Term_p lterm;
    Term_p rterm;
    ...
    struct eqnCell *next; /* For lists of equations */
} EqnCell, *Eqn_p, **EqnRef;
```
Meta-information plus list of literals

typedef struct clause_cell
{
    long ident;            /* Hopefully unique ident for all clauses created during a proof run */
    Eqn_p literals;       /* List of literals */
    FormulaProperties properties;
    Eval_p evaluations;   /* List of evaluations */
    struct clausesetcell* set; /* Is the clause in a set? */
    struct clause_cell* pred; /* For clause sets = doubly */
    struct clause_cell* succ; /* linked lists */
...
}

\[ C_1 = X \not\equiv 0 \lor add(X, s(Y)) \simeq s(Y) \]
$C_1 = X \not\equiv 0 \lor \text{add}(X, s(Y)) \simeq s(Y)$

Clauses

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Id: C1
Eval: 8
...
Clauses

\[ C_1 = ??? \]

Id: C1
...

18
Clauses

\[ C_1 = \Box \]

Id: C1
Eval: 0
...
Saturation Algorithm
\[
\frac{u \not= v \lor R}{\sigma(R)} \quad \text{if} \ \sigma = \text{mgu}(u, v), \quad \ldots
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**Generating inference rules**
- Necessary for completeness
- Increase size of proof state

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**Simplification rules**
- Critical for performance
- Reduce size of proof state

\[
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Requirements for a Saturation Procedure

- Apply all non-redundant generating inferences (in the limit)
  - Necessary for completeness
  - Requires some form of tracking or book-keeping
- Integrate simplification/redundancy elimination
  - Reduces size of proof state and search space
  - Critical for performance
  - Ideal: *No generating inferences involving redundant clauses*
- Support search control
  - No *blind saturation*
  - Suitable choice point(s) for heuristics
- Low overhead
  - Efficient in time
  - Efficient in memory
The Given-Clause Algorithm

We start with a blank slate
We represent the proof state $S$ by two sets of clauses:

- $P$ holds the **processed** clauses (originally empty)
- $U$ holds the **unprocessed** clauses (originally all clauses in $S$)
The Given-Clause Algorithm

Aim: Move everything from $U$ to $P$

$P$ (processed clauses)

$U$ (unprocessed clauses)
The Given-Clause Algorithm

- **Aim:** Move everything from $U$ to $P$

$U$ (unprocessed clauses)

$P$ (processed clauses)

$g = \square$
The Given-Clause Algorithm

Aim: Move everything from $U$ to $P$

Invariant: All generating inferences with premises from $P$ have been performed
The Given-Clause Algorithm

- **Aim:** Move everything from $U$ to $P$
- **Invariant:** All generating inferences with premises from $P$ have been performed
- **Invariant:** $P$ is interreduced

Diagram:
- $P$ (processed clauses)
- $U$ (unprocessed clauses)
- Generate
- Simplify
- Check simplifiability
- $g = \Box$?
The Given-Clause Algorithm

- **Aim**: Move everything from \( U \) to \( P \)
- **Invariant**: All generating inferences with premises from \( P \) have been performed
- **Invariant**: \( P \) is interreduced
- **Clauses added to \( U \) are simplified with respect to \( P \)**
while $U \neq \{\}$
  $g = \text{delete\_best}(U)$
  $g = \text{simplify}(g, P)$
  if $g == \square$
    SUCCESS, Proof found
  if $g$ is not subsumed by any clause in $P$ (or otherwise redundant w.r.t. $P$)
    $P = P \setminus \{c \in P \mid c \text{ subsumed by (or otherwise redundant w.r.t.) } g\}$
    $T = \{c \in P \mid c \text{ can be simplified with } g\}$
    $P = (P \setminus T) \cup \{g\}$
    $T = T \cup \text{generate}(g, P)$
  foreach $c \in T$
    $c = \text{cheap\_simplify}(c, P)$
    if $c$ is not trivial
      $U = U \cup \{c\}$
    SUCCESS, original $U$ is satisfiable
while $U \neq \{}$

$g = \text{delete\_best}(U)$

$g = \text{simplify}(g, P)$

if $g == \square$

SUCCESS, Proof found

if $g$ is not redundant w.r.t. $P$

$P = P \setminus \{c \in P \mid c \text{ redundant w.r.t. } g\}$

$T = \{c \in P \mid c \text{ simplifiable with } g\}$

$P = (P \setminus T) \cup \{g\}$

$T = T \cup \text{generate}(g, P)$

foreach $c \in T$

$c = \text{cheap\_simplify}(c, P)$

if $c$ is not trivial

$U = U \cup \{c\}$

SUCCESS, original $U$ is satisfiable
“You can’t handle the truth!”
Term and Clause Indexing
(Equality Resolution)
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>90% of cpu time
Rewriting

Idea: Replace terms by semantically equal but $\succ$-smaller terms

(Rewriting of negative literals)

\[
\frac{s \simeq t \quad u \not\simeq v \lor R}{s \simeq t \quad u[p \leftarrow \sigma(t)] \not\simeq v \lor R}
\]

if $u|_p = \sigma(s)$ and $\sigma(s) \succ \sigma(t)$

(Rewriting of positive literals)

\[
\frac{s \simeq t \quad u \simeq v \lor R}{s \simeq t \quad u[p \leftarrow \sigma(t)] \simeq v \lor R}
\]

if $u|_p = \sigma(s)$ and $\sigma(s) \succ \sigma(t)$ and $u \simeq v$ is not maximal in $u \simeq v \lor R$ or $u \prec v$ or $p \neq \lambda$
Maximally simplify $g$ with respect to all unit equations in $P$

Maximally simplify all clauses inserted into $U$ with respect to all unit equations in $P$

Check for all clauses in $P$ if they can be simplified with $g$
Simplification

Compute normal form of clause $C$ with respect to $P$:

\[
\textbf{while} \quad \text{C is not in normal form:} \\
\text{for all literals } l \text{ in } C: \\
\quad \text{for all terms } t \text{ in } l: \\
\quad \quad \text{for all subterms } s \text{ of } t: \\
\quad \quad \quad \text{for all unit clauses } l=r \text{ in } P: \\
\quad \quad \quad \quad \sigma = \text{match}(l,s) \\
\quad \quad \quad \quad \quad \text{if } \sigma \text{ and other conditions:} \\
\quad \quad \quad \quad \quad \quad \text{replace } s \text{ by } \sigma(r) \\
\quad \quad \quad \quad \text{else:} \\
\quad \quad \quad \quad \quad \quad \sigma = \text{match}(r,s) \\
\quad \quad \quad \quad \quad \quad \quad \text{if } \sigma \text{ and other conditions:} \\
\quad \quad \quad \quad \quad \quad \quad \quad \text{replace } s \text{ by } \sigma(l)
\]
Simplification

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\[\sigma = \text{match}(r,s)\]
\[\text{if } \sigma \text{ and other conditions}:\]
\[\text{replace } s \text{ by } \sigma(l)\]

...and $|P|$ is growing to $\approx 10^5$ in 300 seconds!
A *term index* is a data structure supporting one or more of the following operations:

- **Given a term** $t$, find all terms $s$ from some set $S$ such that
  - $s$ matches $t$ (our current use case)
    - $t$ is the subterm to be rewritten
      - Every $s$ is a potential left hand side of a unit clause
  - $t$ matches $s$
  - $s$ and $t$ can be unified

- **Indexing can be...**
  - Perfect - all retrieved terms $s$ are in the retrieval relation with the query term $t$
  - Non-perfect - index returns a (hopefully small) superset of candidates
Top Symbol Hashing

- Assume $t \equiv f(t_1, \ldots, t_n)$
- Observation:
  - $\sigma(t) = s$ implies $s \equiv f(s_1, \ldots, s_n)$
  - $t = \sigma(s)$ implies $s \equiv f(s_1, \ldots, s_n)$ or $s \equiv x \in V$
  - $\sigma(t) = \sigma(s)$ implies $s \equiv f(s_1, \ldots, s_n)$ or $s \equiv x \in V$
- ... because $\sigma$ never affects $f$
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Observation:

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... because $\sigma$ never affects $f$

If we organize $P$ by top symbol of potentially maximal sides of unit clauses, we can avoid most matching attempts!
Fingerprint/Discrimination Tree Indexing

- Core idea: Iterated top symbol hashing
  - Traverse tern left-to-right
  - Each function symbol potentially restricts candidates for matching/unification
- Index data structure: Trie
  - Convert terms to symbol strings: \( add(s(X), Y) \implies 'add s X Y' \)
  - Organize strings in a trie
  - Leaf nodes carry original terms and associated date (e.g. originating clause)
- Retrieval:
  - Convert query term into symbols string
  - Follow all compatible branches
  - If a leaf is reached, try candidates stored there
    - For \textit{perfect discrimination trees}, only terms compatible with the retrieval relation will be found
    - For \textit{fingerprint indexing} and \textit{non-perfect discrimination tree indexing}: Candidates must be checked
Impressionist Art

- Non-perfect discrimination tree
- Clause set $P$ from final state of 6th Lusk/Overbeck problem
  - Unit-equational example
  - A ring with $x \times x \times x = x$ is commutative
  - Historically considered hard
  - Now: 0.2 seconds on this computer
Indexing - Summary

- Term indexing can convert the time needed to find inference partners from roughly $O(|P|)$ to $O(\log(|P|))$
- Unification indices speed up paramodulation
  - E.g. Fingerprint Indexing (Choice for E)
  - E.g. Discrimination Tree Indexing (but ugly unification)
- Find-Matching indices speed up forward rewriting
  - E.g. Discrimination Tree Indexing (Choice for E)
  - E.h. Fingerprint Indexing
- Find-Matched indices speed up backwards-rewriting
  - E.g. Fingerprint Indexing (Choice for E)
- Subsumption indices speed up subsumption
  - Index clauses, not terms
  - E.g. Feature Vector Indexing
Search Control
Search Heuristics

- Heuristics are crucial for first-order theorem provers
  - Practical experience is clear
  - Proof search happens in an infinite search space
  - Proofs are rare

- Three major choice points
  - Choice of the term ordering
  - Choice of the literal selection strategy
  - Choice of the next given clause
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The Size of the Problem

How do we make the best choice among millions?
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Choice Point
The Size of the Problem

- $|U| \sim |P|^2$
- $|U| \approx 3 \cdot 10^7$ after 300s
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How do we make the best choice among millions?

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$U$
(unprocessed clauses)
Basic Clause Selection Heuristics

- **Basic idea**: Clauses ordered by heuristic evaluation
  - Heuristic assigns a numerical value to a clause
  - Clauses with smaller (better) evaluations are processed first
- **Example**: Evaluation by symbol counting
  - $|\{f(X) \neq a, P(a) \neq \text{true}, g(Y) = f(a)\}| = 10$
  - Motivation: Small clauses are general, $\square$ has 0 symbols
  - *Best-first* search
- **Example**: FIFO evaluation
  - Clause evaluation based on generation time (always prefer older clauses)
  - Motivation: Simulate *breadth-first* search, find shortest proofs
- **Combine best-first/breadth-first search**
  - E.g. pick 4 out of every 5 clauses according to size, the last according to age
Clause Selection Heuristics in E

- Many symbol-counting variants
  - E.g. Assign different weights to symbol classes (predicates, functions, variables)
  - E.g. Goal directed: lower weight for symbols occurring in original conjecture
  - E.g. ordering-aware/calculus-aware: higher weight for symbols in inference terms

- Arbitrary combinations of base evaluation functions
  - E.g. 5 priority queues ordered by different evaluation functions, weighted round-robin selection
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E can simulate nearly all other approaches to clause selection!
Folklore on Clause Selection/Evaluation

- FIFO is obviously fair, but awful – Everybody
- Preferring small clauses is good – Everybody
- Interleaving best-first (small) and breadth-first (FIFO) is better
  - “The optimal pick-given ratio is 5” – Otter
- Processing all initial clauses early is good – Waldmeister
- Preferring clauses with orientable equation is good – DISCOUNT
- Goal-direction is good – E
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Can we confirm or refute these claims?
Experimental setup

- Prover: E 1.9.1-pre
- 14 different heuristics
  - 13 selected to test folklore claims (interleave 1 or 2 evaluations)
  - Plus modern evolved heuristic (interleaves 5 evaluations)
- TPTP release 6.3.0
  - Only (assumed) provable first-order problems
  - 13774 problems: 7082 FOF and 6692 CNF
- Compute environment
  - StarExec cluster: single threaded run on Xeon E5-2609 (2.4 GHz)
  - 300 second time limit, no memory limit (≥64 GB/core physical)
## Meet the Heuristics

<table>
<thead>
<tr>
<th>Heuristic</th>
<th>Rank</th>
<th>Successes total</th>
<th>Successes unique</th>
<th>Successes within 1s absolute</th>
<th>Successes within 1s of column 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>FIFO</td>
<td>14</td>
<td>4930 (35.8%)</td>
<td>17</td>
<td>3941</td>
<td>79.9%</td>
</tr>
<tr>
<td>SC12</td>
<td>13</td>
<td>4972 (36.1%)</td>
<td>5</td>
<td>4155</td>
<td>83.6%</td>
</tr>
<tr>
<td>SC11</td>
<td>9</td>
<td>5340 (38.8%)</td>
<td>0</td>
<td>4285</td>
<td>80.2%</td>
</tr>
<tr>
<td>SC21</td>
<td>10</td>
<td>5326 (38.7%)</td>
<td>17</td>
<td>4194</td>
<td>78.7%</td>
</tr>
<tr>
<td>RW212</td>
<td>11</td>
<td>5254 (38.1%)</td>
<td>13</td>
<td>5764</td>
<td>79.8%</td>
</tr>
<tr>
<td>2SC11/FIFO</td>
<td>7</td>
<td>7220 (52.4%)</td>
<td>24</td>
<td>5846</td>
<td>79.7%</td>
</tr>
<tr>
<td>5SC11/FIFO</td>
<td>5</td>
<td>7331 (53.2%)</td>
<td>3</td>
<td>5781</td>
<td>78.3%</td>
</tr>
<tr>
<td>10SC11/FIFO</td>
<td>3</td>
<td>7385 (53.6%)</td>
<td>1</td>
<td>5656</td>
<td>77.6%</td>
</tr>
<tr>
<td>15SC11/FIFO</td>
<td>6</td>
<td>7287 (52.9%)</td>
<td>6</td>
<td>5006</td>
<td>82.5%</td>
</tr>
<tr>
<td>GD</td>
<td>12</td>
<td>4998 (36.3%)</td>
<td>12</td>
<td>5856</td>
<td>78.4%</td>
</tr>
<tr>
<td>5GD/FIFO</td>
<td>4</td>
<td>7379 (53.6%)</td>
<td>62</td>
<td>4213</td>
<td>80.2%</td>
</tr>
<tr>
<td>SC11-PI</td>
<td>8</td>
<td>6071 (44.1%)</td>
<td>13</td>
<td>4313</td>
<td>86.3%</td>
</tr>
<tr>
<td>10SC11/FIFO-PI</td>
<td>2</td>
<td>7467 (54.2%)</td>
<td>31</td>
<td>5934</td>
<td>80.4%</td>
</tr>
<tr>
<td>Evolved</td>
<td>1</td>
<td>8423 (61.2%)</td>
<td>593</td>
<td>6406</td>
<td>76.1%</td>
</tr>
</tbody>
</table>
Folklore put to the Test

- FIFO is awful, preferring small clauses is good – mostly confirmed
  - In general, only modest advantage for symbol counting (36% FIFO vs. 39% for best SC)
  - Exception: UEQ (32% vs. 63%)

- Interleaving best-first/breadth-first is better – confirmed
  - 54% for interleaving vs. 39% for best SC

- Influence of different pick-given ratios is surprisingly small
  - UEQ is again an outlier (60% for 2:1 vs. 70% for 15:1)

- The optimal pick-given ratio is 10 (for E)

- Processing all initial clauses early is good – confirmed
  - Effect is less pronounced for interleaved heuristics

- Preferring clauses with orientable equation is good – not confirmed
  - There is no evidence in our data, not even for UEQ

- Goal-direction is good – partially confirmed
  - GD on its own performs similar to SC
  - GD shines in combination with FIFO
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Selected Results

▶ Good heuristics do make a difference
  ► 71% more solutions with Evolved vs. FIFO
  ► 58% more solutions with Evolved vs. best SC

... with little variation between strategies (spread: 76%–84%)

Success comes early
≈ 80% of all proofs found in less than 1s

Cooperation beats portfolio/strategy scheduling

SC11 solves 5340 problems
FIFO solves 4930 problems
Union of the previous two contains 6329 problems
... but 10SC11/FIFO solves 7385

Evolving paid off

... significantly better than best "naive" heuristic

10 × more unique solutions than second-best
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Bonus Material
Small things...

- Version control (git and beyond)
  - Value of version control should be self-evident
  - But: Have the system *print* a version number
    - Automatically record the version number with experiments
    - Loudly complain about bug reports without version number (and command line and input file)

- Automatic input format
  - E supports TPTP-3 (TFA/FOF/CNF), TPTP-2 and LOP
    - Originally only selectable by option (e.g. --tptp3-in)
    - Now automatic default based on first token
  - Automatic format is a big win for convenience
90% of the Iceberg is Under Water

Preprocessor

Control

Proof extraction

Support Tools

(Python, bash, gawk)

Logical data types

Generic data types

Low level support code

Language API/Libraries

Operating System (POSIX)
Alternative Term Implementations

- **Shared terms (e.g. in E)**
  - Structure as for normal terms
  - Only one copy of structurally identical terms \((f(a, a)\) has two nodes)
  - Can save 80% to 99.9% of term nodes
  - Can be used to cache values/results
  - Can use garbage collection (even in C) - saves a lot of time and headaches!

- **Flat terms (e.g. in Waldmeister, Twee)**
  - Term is flat string of symbol encoding
  - Very space-efficient
  - Very fast traversal for matching
  - Harder to manipulate
Implementing Term Orderings

- **Lexicographic Path Ordering (LPO)**
  - Based on function symbol precedence
  - Top symbols and symbol bags decide
  - Lexicographic decomposition for identical top symbols
  - LPO has theoretical advantages
    - Can orient equations towards sides with more variable occurrences
    - Can orient distributively *the right way* with $\times > +$:
      \[ X \times (Y + Z) \rightarrow (X \times Y) + (X \times Z) \]

- **Knuth-Bendix-Ordering (KBO)**
  - Based on symbols weights and precedence
  - Weight, top symbol, decomposition, *variable condition*
  - KBO has practical advantages
    - Orient towards syntactically small terms (keeps terms smaller)
    - More efficient to evaluate

- For efficient algorithms: Löchner’s *Things to Know When Implementing [KL]PO* [5, 6]
Conclusion
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- Building a basic saturating prover is not hard
  - Binary resolution + factoring
  - Unification
  - (Subsumption)

- Building a superposition prover is harder
  - Term orderings
  - Rewriting / Simplification

- Being competitive is harder still
  - Indexing
  - Search heuristics
  - Experimental infrastructure

The knowledge needed to write a decent prover is now mostly publicly available!
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- Maintenance and evolution are hardest!
  - Users want new features
  - Conference/competition deadline always loom
  - Result: Code clutter
  - Unexpected dependencies and side effects

- . . . so keep it clean!
  - Have a plan
  - Code clean and general building blocks
  - Take the time to refactor/cleanup (yeah, right!)
  - . . . or throw it away and start over

- Remember: 90% of the iceberg . . .
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