Implementation of First-Order Theorem Provers

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First-Order Theorem Proving

Given: A set axioms and a hypothesis in first-order logic

$$A = \{A_1, \ldots, A_n\},\ H$$

Question: Do the axioms logically imply the hypothesis?

$$A \overset{?}{\models} H$$

An automated theorem prover tries to solve this question!

First-Order Logic with Equality

- First order logic deals with
  - Elements
  - Relations between elements
  - Functions over elements
  - . . . and their combination

- Allows general statements using quantified variables
  - There exists an X so that property P holds ($\exists X : P(X)$)
  - For all possible values of X property P holds ($\forall X : P(X)$)

- Function and predicate symbols are uninterpreted
  - No implicit background theory
  - All properties have to be specified explicitely
  - Exception: Equality is interpreted (as a congruence relation)

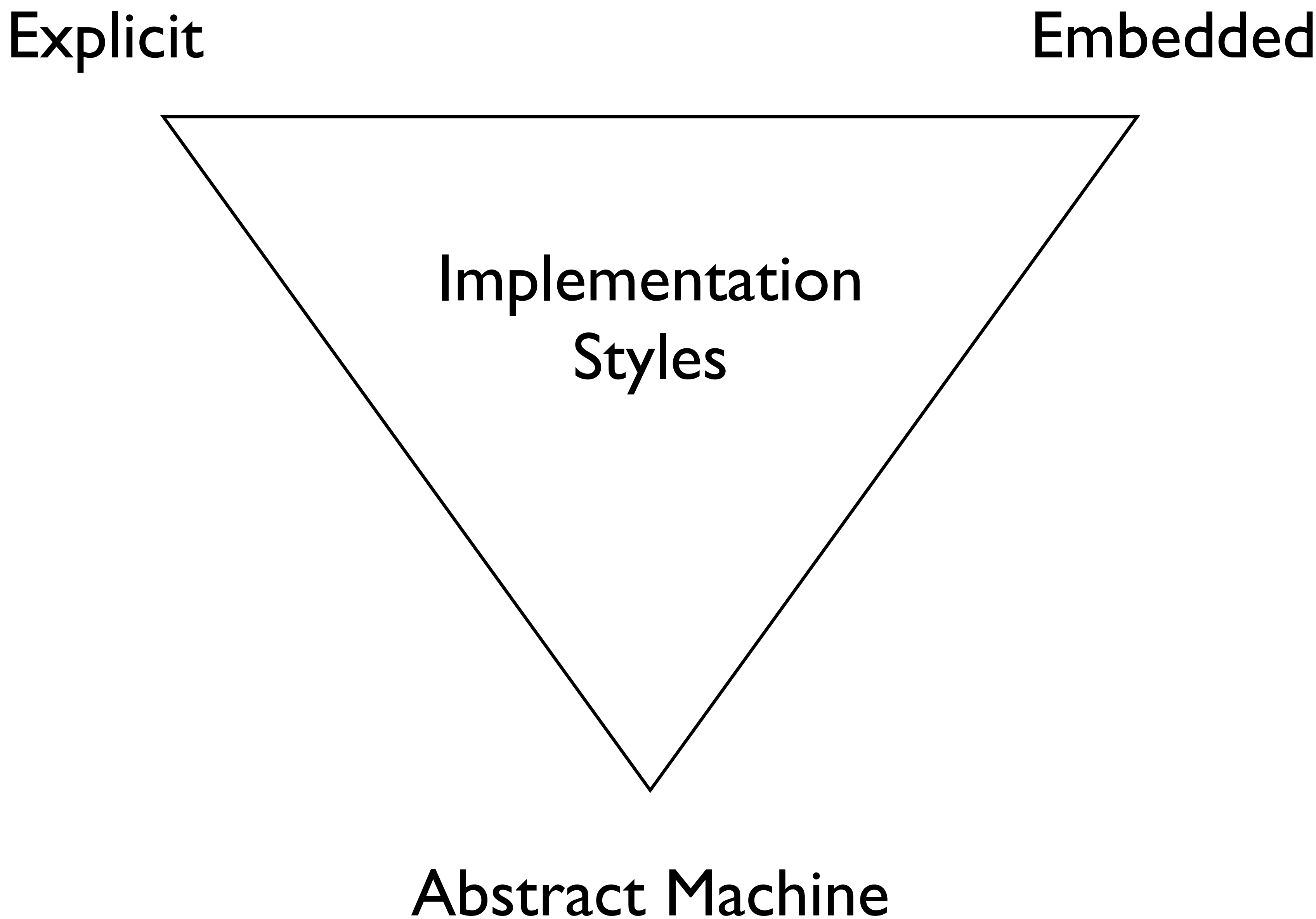
Why First-Order Logic?

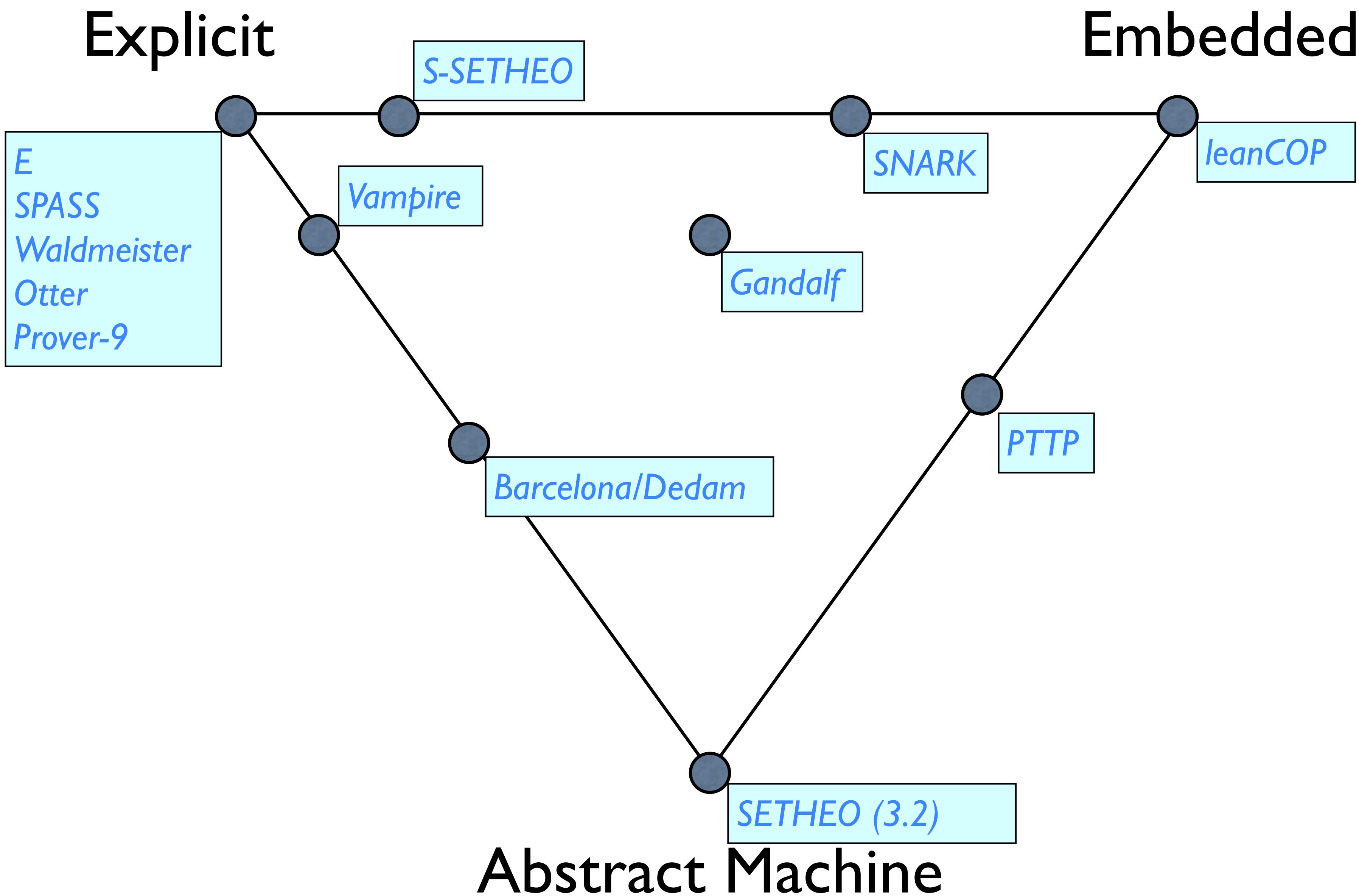
- Expressive:
  - Can encode any computable problem
  - Most tasks can be specified reasonably naturally
  - Many other logics can be reasonably translated to first-order logic
- Automatizable:
  - Sound and complete calculi for proof search exist
  - Search procedures are reasonably efficient
- Stable:
  - Logic is well-known and well-understood
  - Semantics are clear (and somewhat intuitive)

First-order logic is a good compromise between expressiveness and automatizability

Mainstream Milestones

- Herbrand-Universe Enumeration+SAT [DP60]
- Resolution [Rob65]
- Model Elimination [Lov68]
- Paramodulation [RW69]
- Completion [KB70]
- Otter 1.0 (1989, McCune)
- Unfailing completion [BDP89, HR87]
- Superposition [BG90, NR92, BG94]
- SETHEO [LSBB92]
- Vampire [Vor95] (but kept hidden for years)
- First CASC competition at Rutgers, FLOC'96 (Sutcliffe, Suttner)
- Waldmeister [BH96]
- SPASS [WGR96]
- E [Sch99]


Explicit
Embedded
Implementation
Styles
Abstract Machine


Explicit
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S-SETHEO
E
SPASS
Waldmeister
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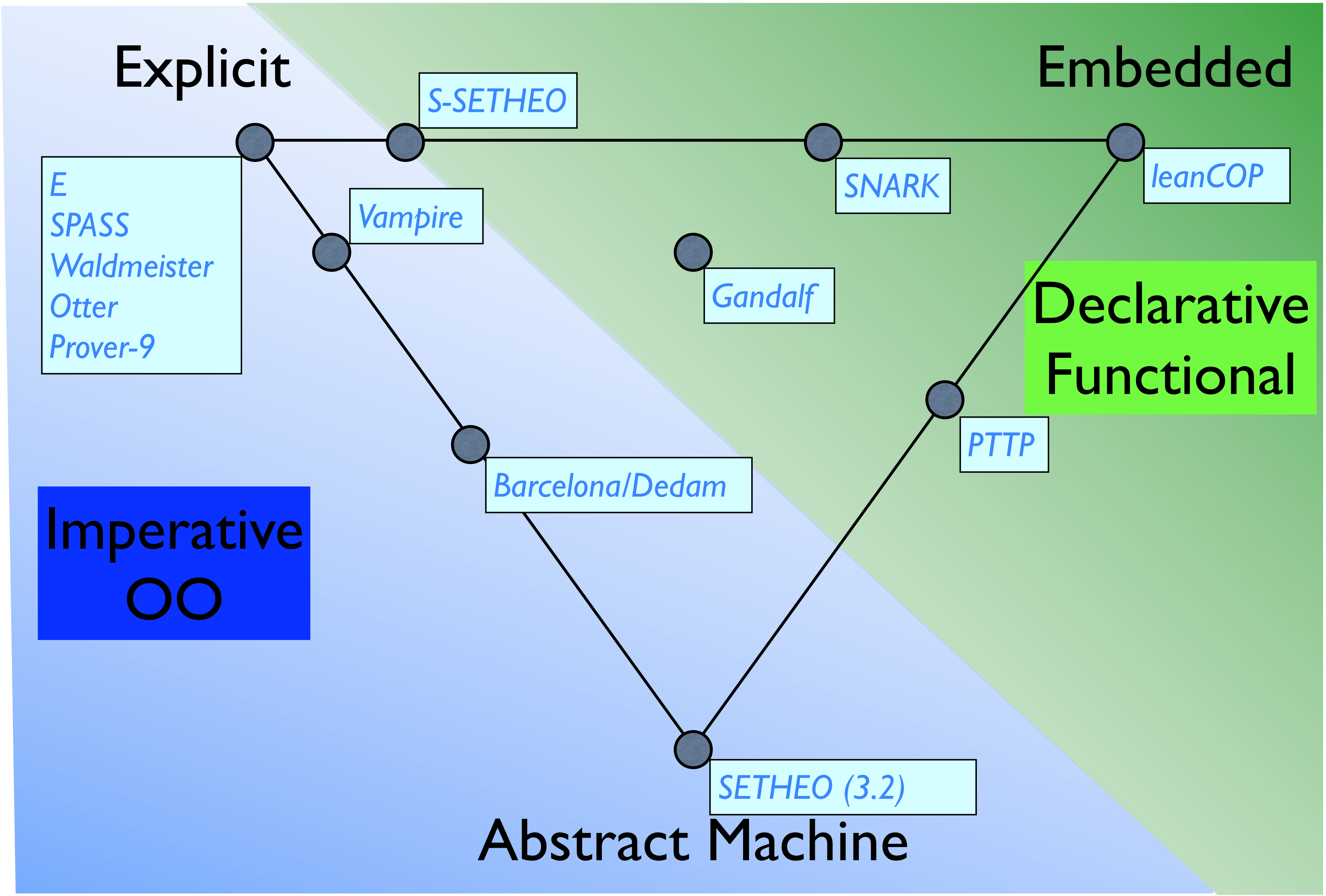
Gandalf
PTTP
Barcelona/Dedam
SETHEO (3.2)
Abstract Machine


Explicit
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Declarative
Functional
PTTP
Barcelona/Dedam
Imperative
OO
SETHEO (3.2)
Abstract Machine

Implementation Style (References)

Barcelona/Dedam [NRV97]
Gandalf [Tam97]
PTTP [Sti92, Sti89]
S-SETHEO [LS01b]
SPASS [Wei01, WSH$^+$07]
Vampire [RV02]
leanCOP [OB03, Ott08]
E [Sch02, Sch04b]
Otter [MW97]
Prover-9 [McC08]
SETHEO [LSBB92, MIL$^+$97]
Snark [E.S08]
Waldmeister [LH02, GHLS03]

Formulae

- Formulas are recursively defined:
  - Literals (elementary statements) are formulae
  - If F is a formula, $\forall X : F$ and $\exists X : F$ are formulae
  - Boolean combinations of formulae are formulae
  - Parentheses are applied wherever necessary
- Example:
  - $\forall X : (\forall Y : ((odd(X) \land odd(Y)) \rightarrow X \not\simeq add(Y, 1)))$

Clauses

- Clauses are multisets written and interpreted as disjunctions of literals
  - All variables implicitly universally quantified
- Example:

$$X \not\simeq add(Y,1) \vee odd(X) \vee odd(Y)$$

- Alternative views: Implicational

$$X \simeq add(Y,1) \implies (odd(X) \vee odd(Y))$$

or

$$(X \simeq add(Y,1) \wedge \neg odd(X)) \implies odd(Y))$$

or

$$(X \simeq add(Y,1) \wedge \neg odd(Y)) \implies odd(X))$$

or (weirdly)

$$(\neg odd(Y) \wedge \neg odd(X)) \implies X \not\simeq add(Y,1)$$

Literals

- $X \not\simeq add(Y, 1) \vee odd(X) \vee odd(Y)$

- – $X \not\simeq add(Y, 1)$ is a negative equational literal
  – $odd(X)$ and $odd(X)$ are positive non-equational literals

- Conventions:

  – $s \not\simeq t$ is a more convenient way of writing $\neg s \simeq t$
  – We write $s \dot{\simeq} t$ to denote an equational literal that may be either positive or negative
  – $s \simeq t$ is a more convenient way of writing $\simeq (s, t)$

Literals

- $X \not\simeq add(Y, 1) \vee odd(X) \vee odd(Y)$

- – $X \not\simeq add(Y, 1)$ is a negative equational literal
  – $odd(X)$ and $odd(X)$ are positive non-equational literals

- Convention:

  – $s \not\simeq t$ is a more convenient way of writing $\neg s \simeq t$
  – We write $s \dot{\simeq} t$ to denote an equational literal that may be either positive or negative
  – Heresy: $s \simeq t$ is a more conventient way of writing $\simeq (s, t)$
  – Truth: $odd(X)$ is a more convenient way of writing $odd(X) \simeq \top$

Equational Encoding Snag

- Problem:
  - $\{X \simeq a), \neg p(a)\}$ is satisfiable
  - What about $\{X \simeq a), p(a) \not\simeq \top\}$?
- Solution:
  - Two sorts: Individuals and Bools
  - Variables range over individuals only
  - Predicate terms are sort Bool
- Implemented that way in E

Terms

- $X \not\simeq add(Y, 1) \vee odd(X) \vee odd(Y)$

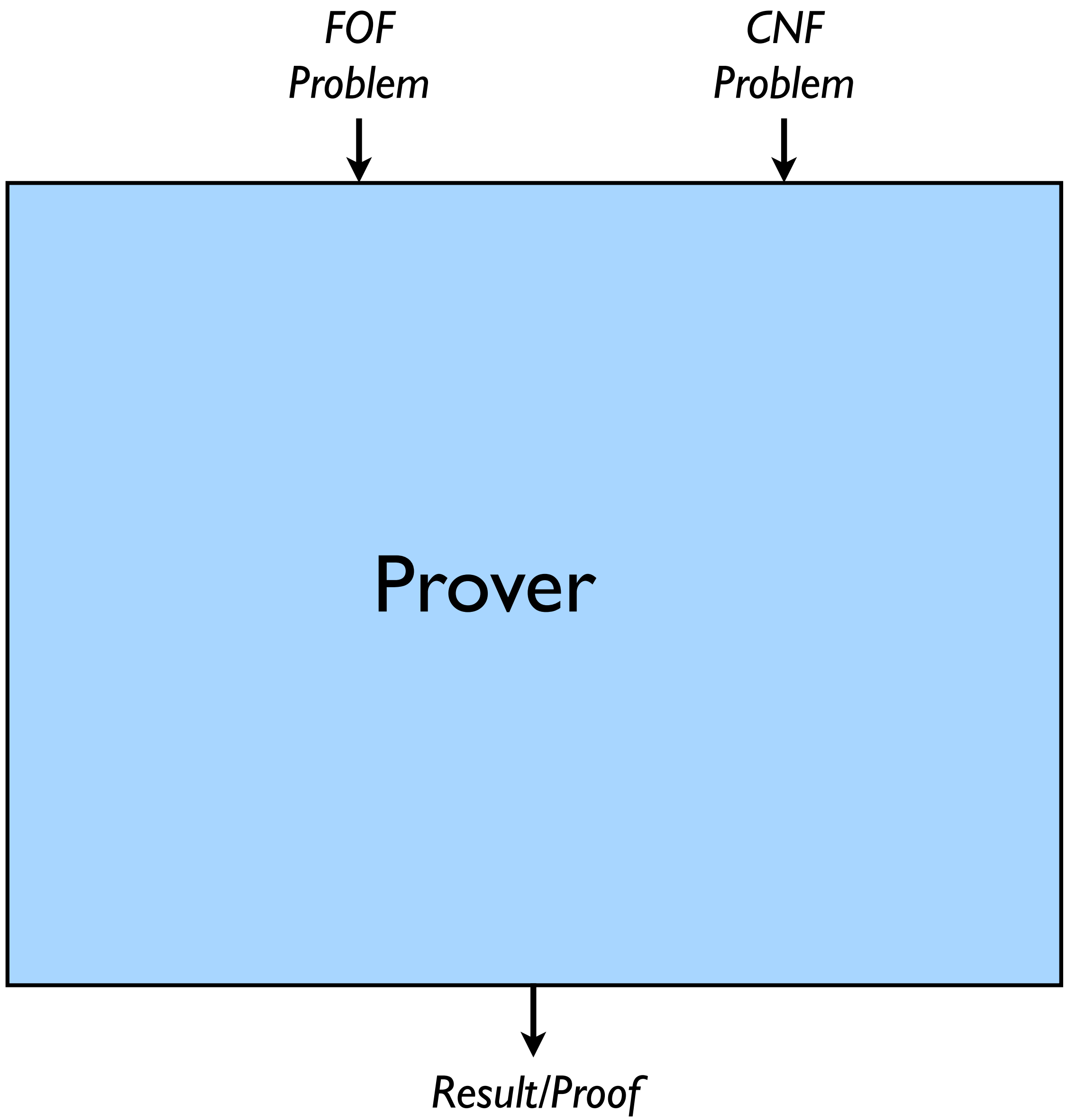
- – $X, add(Y, 1), 1$, and Y are terms
  – X and Y are variables
  – 1 is a constant term
  – $add(Y, 1)$ is a composite term with proper subterms 1 and Y

Concrete Syntax

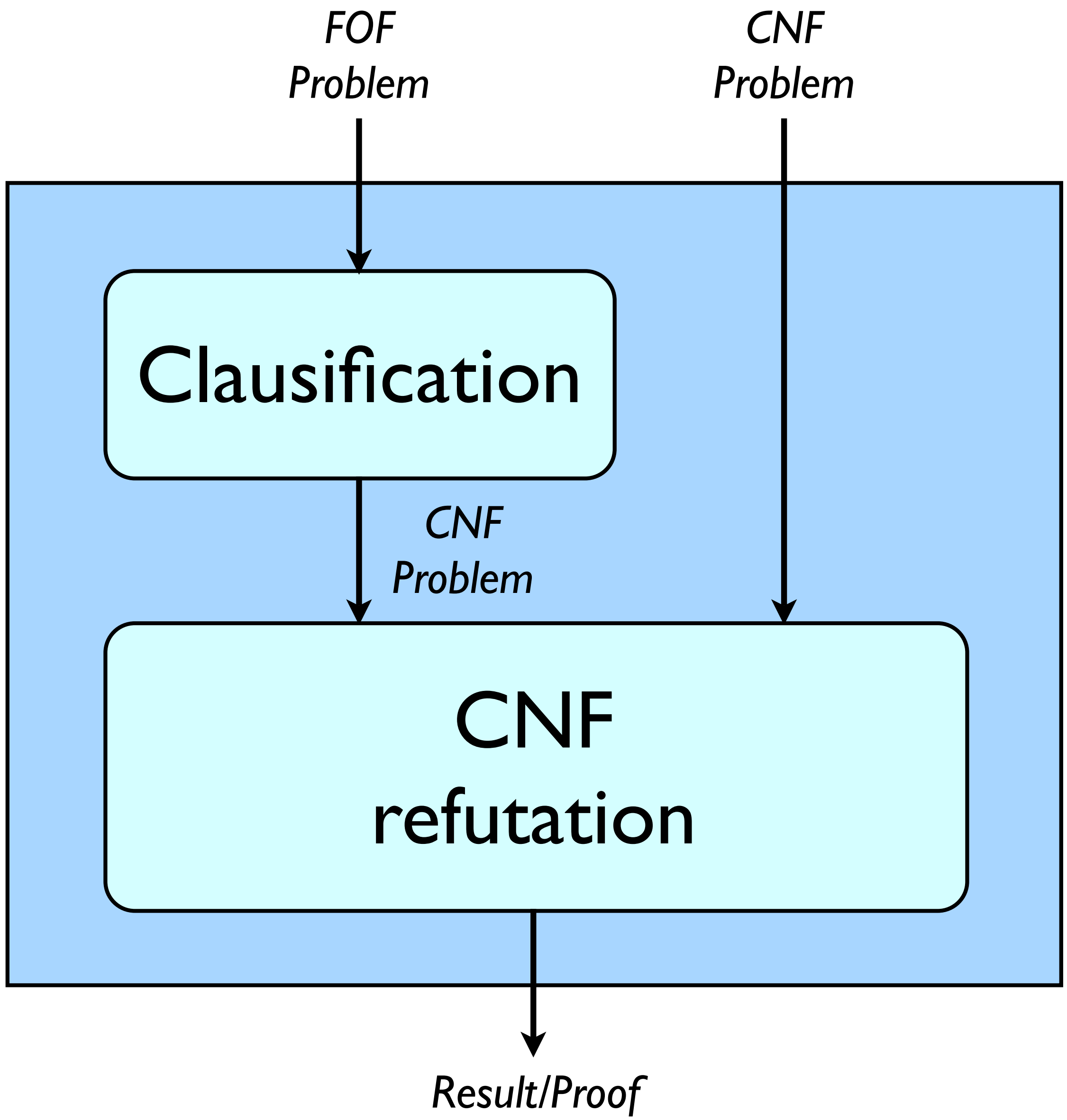
- Historically: Large variety of syntaxes
  - Prolog-inspired, e.g. LOP (SETHEO, E)
  - By committee, e.g. DFG-Syntax (SPASS)
  - LISP-inspired (SNARK)
  - Home-grown (Otter, Prover-9)
  - TPTP-1/2 syntax (with TPTP2X converter)

- Recently: Quasi-standardizaton on TPTP-3 syntax [SSCG06, Sut09]
  - Annotated clauses/formulas
  - Can represent problems and proofs
  - Support in Vampire, SPASS, E, E-SETHEO, iProver,

A First-Order Prover - Bird's Eye Perspective



A First-Order Prover - Bird's X-Ray Perspective



Clausification

$A \stackrel{?}{\models} H$ $\quad\Longrightarrow\quad$ $\boxed{\textbf{Classifier}}$ $\quad\Longrightarrow\quad$ $\{C_1, C_2, \ldots, C_3\}$

...such that

$\{C_1, C_2, \ldots, C_3\}$ is unsatisfiable

iff

$A \models H$ holds

Clausification

$A \overset{?}{\models} H \quad \Longrightarrow \quad \boxed{\textbf{Magic}} \quad \Longrightarrow \quad \{C_1, C_2, \ldots, C_3\}$

...such that

$\{C_1, C_2, \ldots, C_3\}$ is unsatisfiable

iff

$A \models H$ holds

Clausification

$A \overset{?}{\models} H \quad \Longrightarrow \quad \boxed{\textbf{Magic}} \quad \Longrightarrow \quad \{C_1, C_2, \ldots, C_3\}$

White Magic: Standard conjunctive normal form with Skolemization [Lov78] [NW01] (read once)

- Straightforward
- CNF can explode (and does, occasionally)

Black Magic: Miniscoping and definitions [NW01] (Read twice)

- Smaller CNF, exponential growths can be controlled
- Better (smaller) terms, less arity in Skolem functions
- Implemented in **E**

Forbidden Magic: Advanced Skolemization [NW01](Read five times)

- Implemented in FLOTTER
- Theoretically superior, but advantage in practice unclear

Why FOF at all?

```
% All aircraft are either in lower or in upper airspace
fof(low_up_is_exhaustive, axiom,
    (![X]:(lowairspace(X)|uppairspace(X)))).

fof(filter_equiv, conjecture, (
% Naive version: Display aircraft in the Abu Dhabi Approach area in
% lower airspace, display aircraft in the Dubai Approach area in lower
% airspace, display all aircraft in upper airspace, except for
% aircraft in military training region if they are actual military
% aircraft.
    (![X]:(((a_d_app(X) & lowairspace(X))|(dub_app(X) & lowairspace(X))
    |uppairspace(X))&
    (~milregion(X)|~military(X))))
    <=>
% Optimized version: Display all aircraft in either Approach, display
% aircraft in upper airspace, except military aircraft in the military
% training region
    (![X]:((uppairspace(X) | dub_app(X) | a_d_app(X)) &
    (~military(X) | ~milregion(X)))))).
```

Why FOF at all?

```
cnf(i_0_1,plain,(lowairspace(X1)|uppairspace(X1))).
cnf(i_0_12,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~uppairspace(esk1_0)|~uppairspace(esk2_0))).
cnf(i_0_8,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~uppairspace(esk1_0)|~a_d_app(esk2_0))).
cnf(i_0_10,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~uppairspace(esk1_0)|~dub_app(esk2_0))).
cnf(i_0_13,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~uppairspace(esk1_0)|~uppairspace(esk2_0))).
cnf(i_0_9,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~uppairspace(esk1_0)|~a_d_app(esk2_0))).
cnf(i_0_11,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~uppairspace(esk1_0)|~dub_app(esk2_0))).
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cnf(i_0_2,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~uppairspace(esk1_0)|~a_d_app(esk2_0))).
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cnf(i_0_5,negated_conjecture,(military(esk1_0)|military(esk2_0)|~uppairspace(esk1_0)|~dub_app(esk2_0))).
cnf(i_0_36,negated_conjecture,(milregion(esk1_0)|milregion(esk2_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
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    ~dub_app(esk1_0))).
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    ~dub_app(esk2_0))).
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    ~a_d_app(esk1_0))).
cnf(i_0_25,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
    ~dub_app(esk1_0))).
cnf(i_0_33,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk1_0)|
    ~a_d_app(esk2_0))).
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cnf(i_0_35,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~a_d_app(esk1_0)|
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    ~dub_app(esk1_0))).
cnf(i_0_23,negated_conjecture,(milregion(esk1_0)|military(esk2_0)|~lowairspace(esk1_0)|~dub_app(esk1_0)|
    ~dub_app(esk2_0))).
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    ~a_d_app(esk1_0))).
cnf(i_0_18,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~lowairspace(esk1_0)|~uppairspace(esk2_0)|
    ~dub_app(esk1_0))).
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    ~dub_app(esk2_0))).
cnf(i_0_14,negated_conjecture,(milregion(esk2_0)|military(esk1_0)|~lowairspace(esk1_0)|~a_d_app(esk2_0)|
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    ~a_d_app(esk2_0))).
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    ~dub_app(esk2_0))).
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    dub_app(X1))).
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    dub_app(X1))).
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cnf(i_0_43,negated_conjecture,(uppairspace(X1)|a_d_app(X1)|dub_app(X1)|~milregion(X2)|~military(X2))).
cnf(i_0_38,negated_conjecture,(~milregion(X2)|~milregion(X1)|~military(X2)|~military(X1))).
```
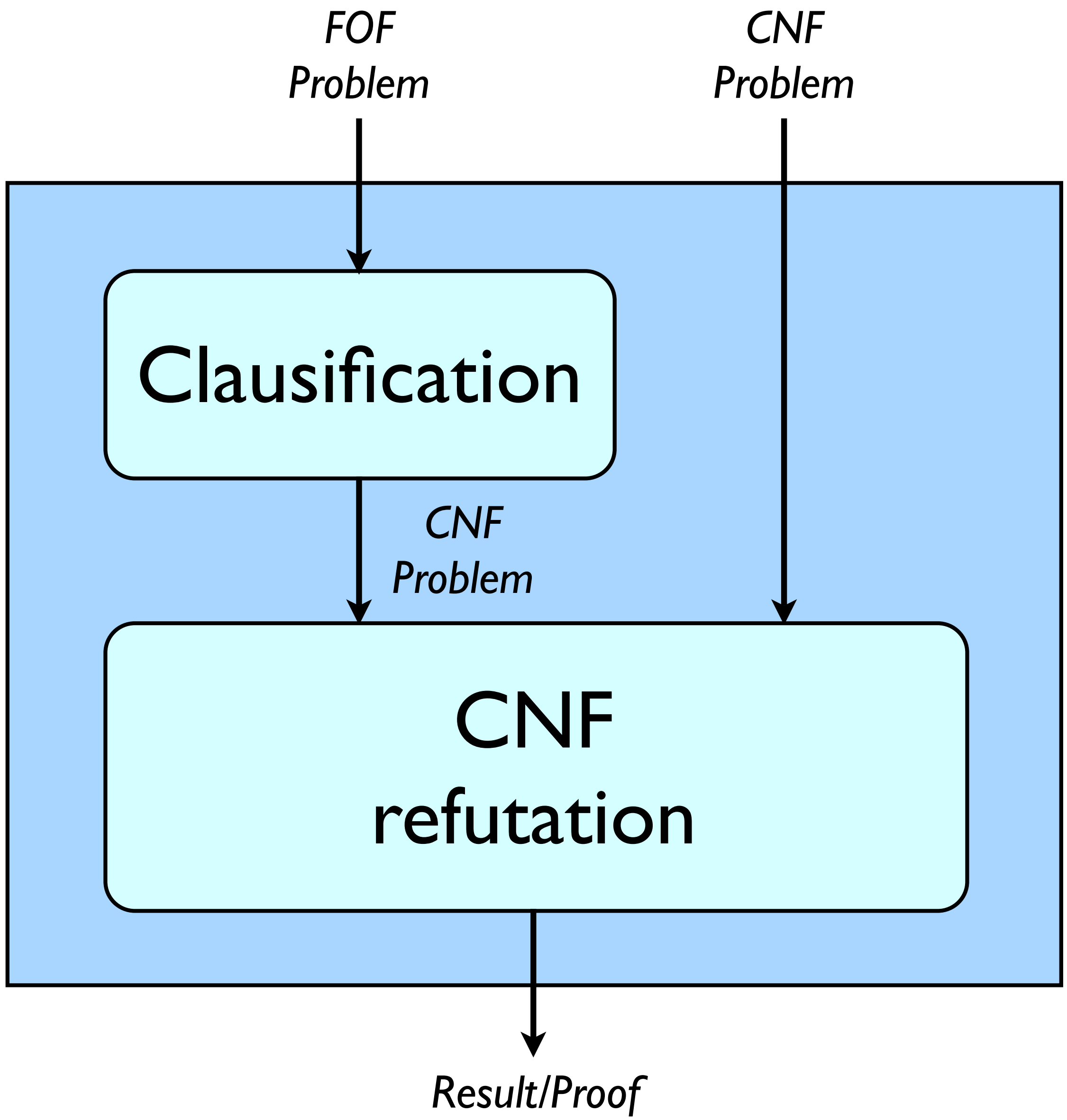

Lazy Developer's Clausification

$$A \overset{?}{\models} H \quad \Longrightarrow \quad \boxed{\begin{array}{c}\textbf{E}\\ \textbf{FLOTTER}\\ \textbf{Vampire}\end{array}} \quad \Longrightarrow \quad \{C_1, C_2, \ldots, C_3\}$$

- iProver (uses E, Vampire)
- E-SETHEO (uses E, FLOTTER)
- Fampire (uses FLOTTER)

A First-Order Prover - Bird's X-Ray Perspective



CNF Saturation

- Basic idea: Proof state is a set of clauses S
  - Goal: Show unsatisfiability of S
  - Method: Derive empty clause via deduction
  - Problem: Proof state explosion
- Generation: Deduce new clauses
  - Logical core of the calculus
  - Necessary for completeness
  - Lead to explosion is proof state size
    $\Longrightarrow$ Restrict as much as possible
- Simplification: Remove or simplify clauses from S
  - Critical for acceptable performance
  - Burns most CPU cycles
    $\Longrightarrow$ Efficient implementation necessary

Rewriting

- Ordered application of equations
  - Replace equals with equals. . .
  - . . . if this decreases term size with respect to given ordering $>$

$$\frac{s \simeq t \qquad u \dot{\simeq} v \vee R}{s \simeq t \qquad u[p \leftarrow \sigma(t)] \dot{\simeq} v \vee R}$$

- Conditions:
  - $u|_p = \sigma(s)$
  - $\sigma(s) > \sigma(t)$
  - Some restrictions on rewriting $>$-maximal terms in a clause apply

- Note: If $s > t$, we call $s \simeq t$ a rewrite rule
  - Implies $\sigma(s) > \sigma(t)$, no ordering check necessary

Paramodulation/Superposition

- Superposition: "Lazy conditional speculative rewriting"
  - Conditional: Uses non-unit clauses
    - One positive literal is seen as potential rewrite rule
    - All other literals are seen as (positive and negative) conditions
  - Lazy: Conditions are not solved, but appended to result
  - Speculative:
    - Replaces potentially larger terms
    - Applies to instances of clauses (generated by unification)
    - Original clauses remain (generating inference)

$$\frac{s \simeq t \vee S \quad u \dot{\simeq} v \vee R}{\sigma(u[p \leftarrow t] \dot{\simeq} v \vee S \vee R)}$$

- Conditions:
  - $\sigma = mgu(u|_p, s)$ and $u|_p$ is not a variable
  - $\sigma(s) \not< \sigma(t)$ and $\sigma(u) \not< \sigma(v)$
  - $\sigma(s \simeq t)$ is $>$-maximal in $\sigma(s \simeq t \vee S)$ (and no negative literal is selected)
  - $\sigma(u \dot{\simeq} v)$ is maximal (and no negative literal is selected) or selected

Subsumption

- Idea: Only keep the most general clauses
  - If one clause is subsumed by another, discard it

$$\frac{C \qquad \sigma(C) \vee R}{C}$$

- Examples:
  - $p(X)$ subsumes $p(a) \vee q(f(X), a)$ ($\sigma = \{X \leftarrow a\}$)
  - $p(X) \vee p(Y)$ does not multi-set-subsume $p(a) \vee q(f(X), a)$
  - $q(X, Y) \vee q(X, a)$ subsumes $q(a, a) \vee q(a, b)$

- Subsumption is hard (NP-complete)
  - $n!$ permutations in non-equational clause with n literals
  - $n!2^n$ permutations in equational clause with n literals

Term Orderings

- Superposition is instantiated with a ground-completable simplification ordering $>$ on terms
  - $>$ is Noetherian
  - $>$ is compatible with term structure: $t_1 > t_2$ implies $s[t_1]_p > s[t_2]_p$
  - $>$ is compatible with substitutions: $t_1 > t_2$ implies $\sigma(t_1) > \sigma(t_2)$
  - $>$ has the subterm-property: $s > s|_p$
  - In practice: LPO, KBO, RPO
- Ordering evaluation is one of the major costs in superposition-based theorem proving
- Efficient implementation of orderings: [Löc06, L0̈6]

Generalized Redundancy Elimination

- A clause is redundant in S, if all its ground instances are implied by $>$ smaller ground instances of other clauses in S
  - May require addition of smaller implied clauses!
- Examples:
  - Rewriting (rewritten clause added!)
  - Tautology deletion (implied by empty clause)
  - Redundant literal elimination: $l \vee l \vee R$ replaced by $l \vee R$
  - False literal elimination: $s \not\simeq s \vee R$ replaced by R
- Literature:
  - Theoretical results: [BG94, BG98, NR01]
  - Some important refinements used in E: [Sch02, Sch04b, RV01, Sch09]

The Basic Given-Clause Algorithm

- Completeness requires consideration of all possible persistent clause combinations for generating inferences
  - For superposition: All 2-clause combinations
  - Other inferences: Typically a single clause
- Given-clause algorithm replaces complex bookkeeping with simple invariant:
  - Proofstate $S = P \cup U$, P initially empty
  - All inferences between clauses in P have been performed
- The algorithm:

while $U \neq \{\}$
  $g =$ delete_best(U)
  if $g == \Box$
    SUCCESS, Proof found
  $P = P \cup \{g\}$
  $U = U \cup$generate(g, P)
SUCCESS, original U is satisfiable

DISCOUNT Loop

- Aim: Integrate simplification into given clause algorithm
- The algorithm (as implemented in E):

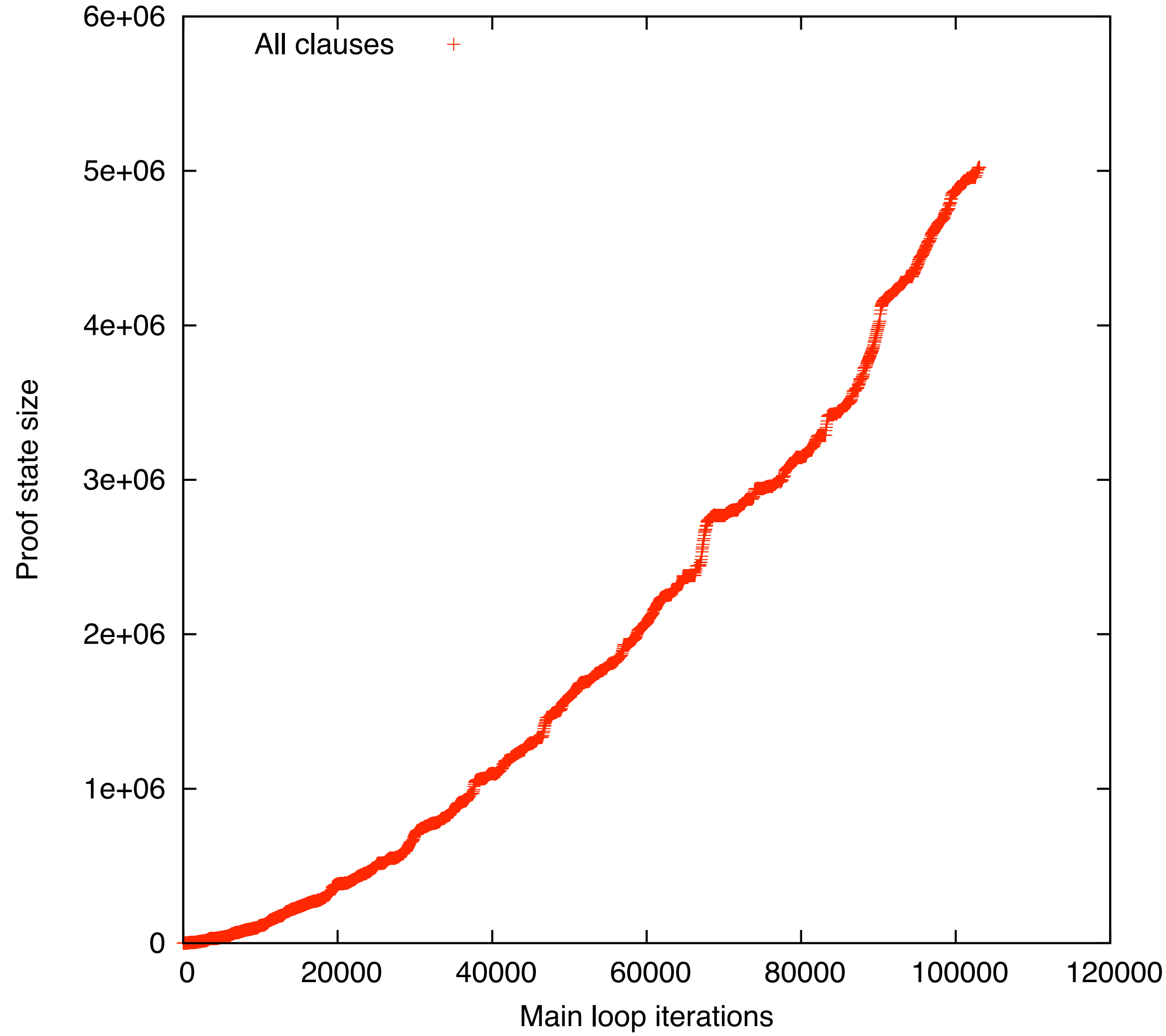
while $U \neq \{\}$
  $g =$ delete_best(U)
  $g =$ simplify(g,P)
  if $g == \square$
    SUCCESS, Proof found
  if g is not redundant w.r.t. P
    $T = \{c \in P | c$ redundant or simplifiable w.r.t. $g\}$
    $P = (P \backslash T) \cup \{g\}$
    $T = T \cup$generate(g, P)
    foreach $c \in T$
      $c =$cheap_simplify(c, P)
      if c is not trivial
        $U = U \cup \{c\}$
SUCCESS, original U is satisfiable

What is so hard about this?

What is so hard about this?

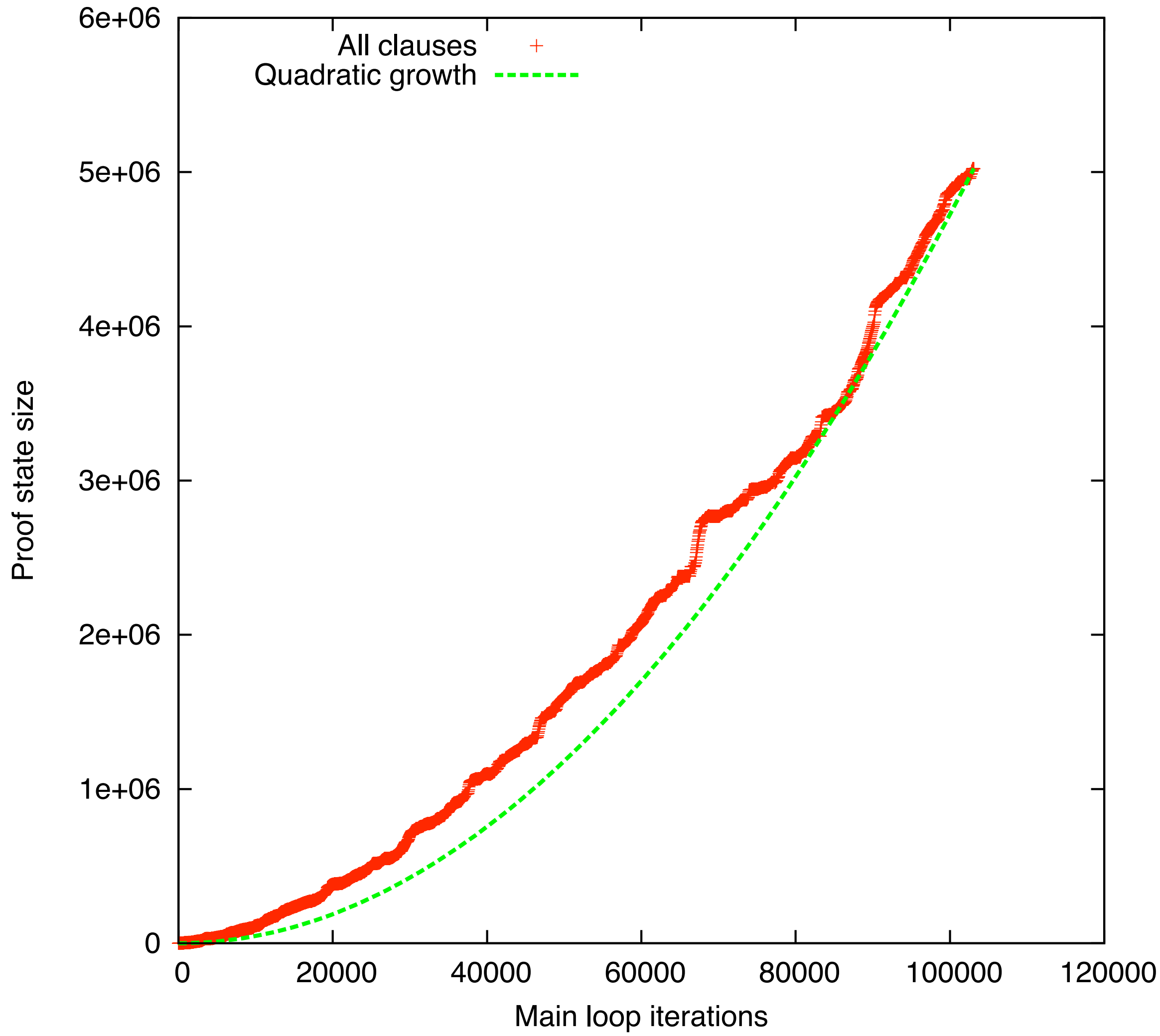
- Data from simple TPTP example NUM030-1+rm_eq_rstfp.lop (solved by E in 30 seconds on ancient Apple Powerbook):
  - Initial clauses: 160
  - Processed clauses: 16,322
  - Generated clauses: 204,436
  - Paramodulations: 204,395
  - Current number of processed clauses: 1,885
  - Current number of unprocessed clauses: 94,442
  - Number of terms: 5,628,929
- Hard problems run for days!
  - Millions of clauses generated (and stored)
  - Many millions of terms stored and rewritten
  - Each rewrite attempt must consider many ($>>$ 10000) rules
  - Subsumption must test many ($>>$ 10000) candidates for each subsumption attempt
  - Heuristic must find best clause out of millions

Proof State Development



Proof state behavior for ring theory example RNG043-2 (Default Mode)

Proof State Development



Proof state behavior for ring theory example RNG043-2 (Default Mode)

- Growth is roughly quadratic in the number of processed clauses

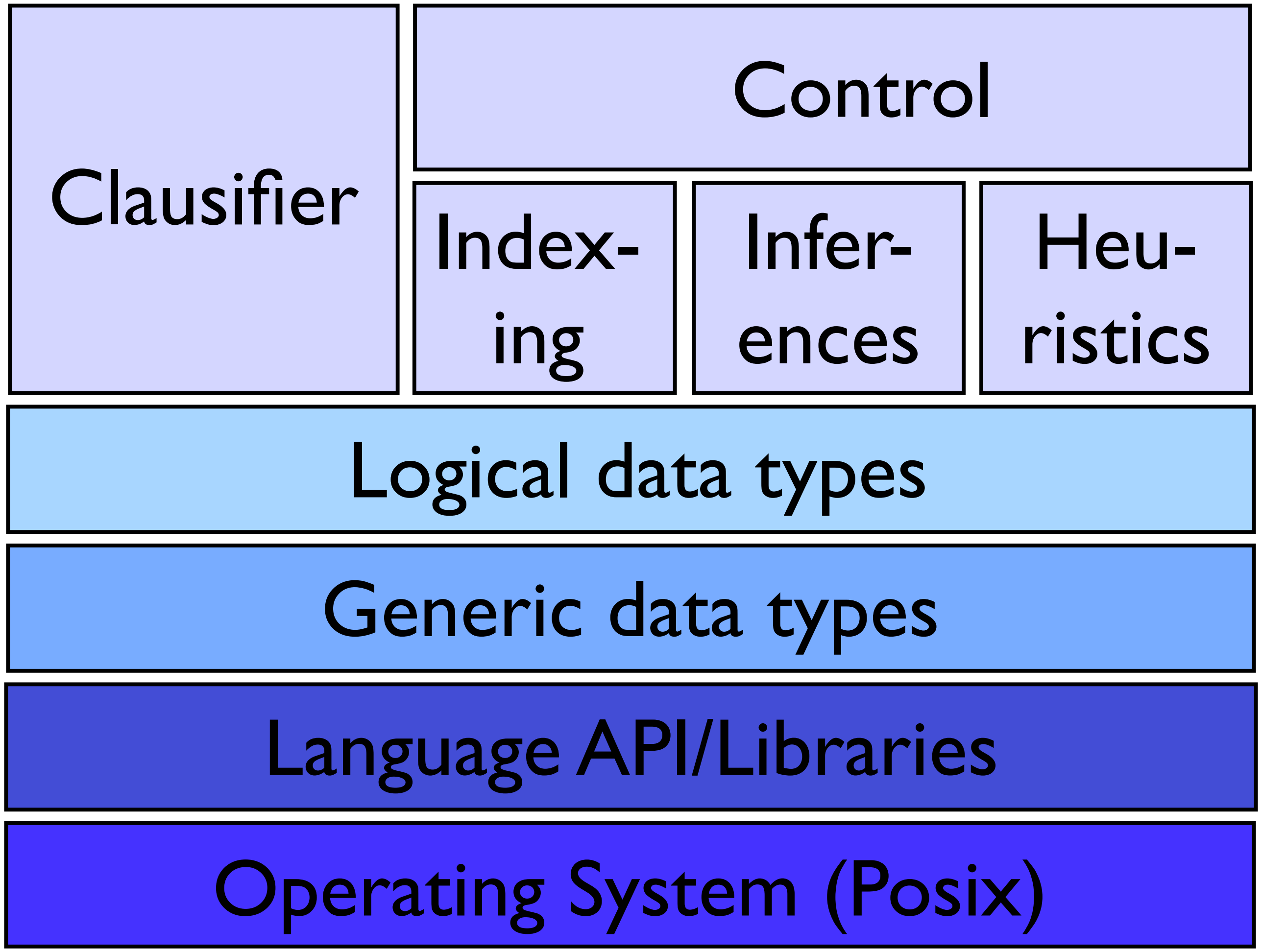
Literature on Proof Procedures

- New Waldmeister Loop: [GHLS03]
- Comparisons: [RV03]
- Best discussion of E Loop: [Sch02]

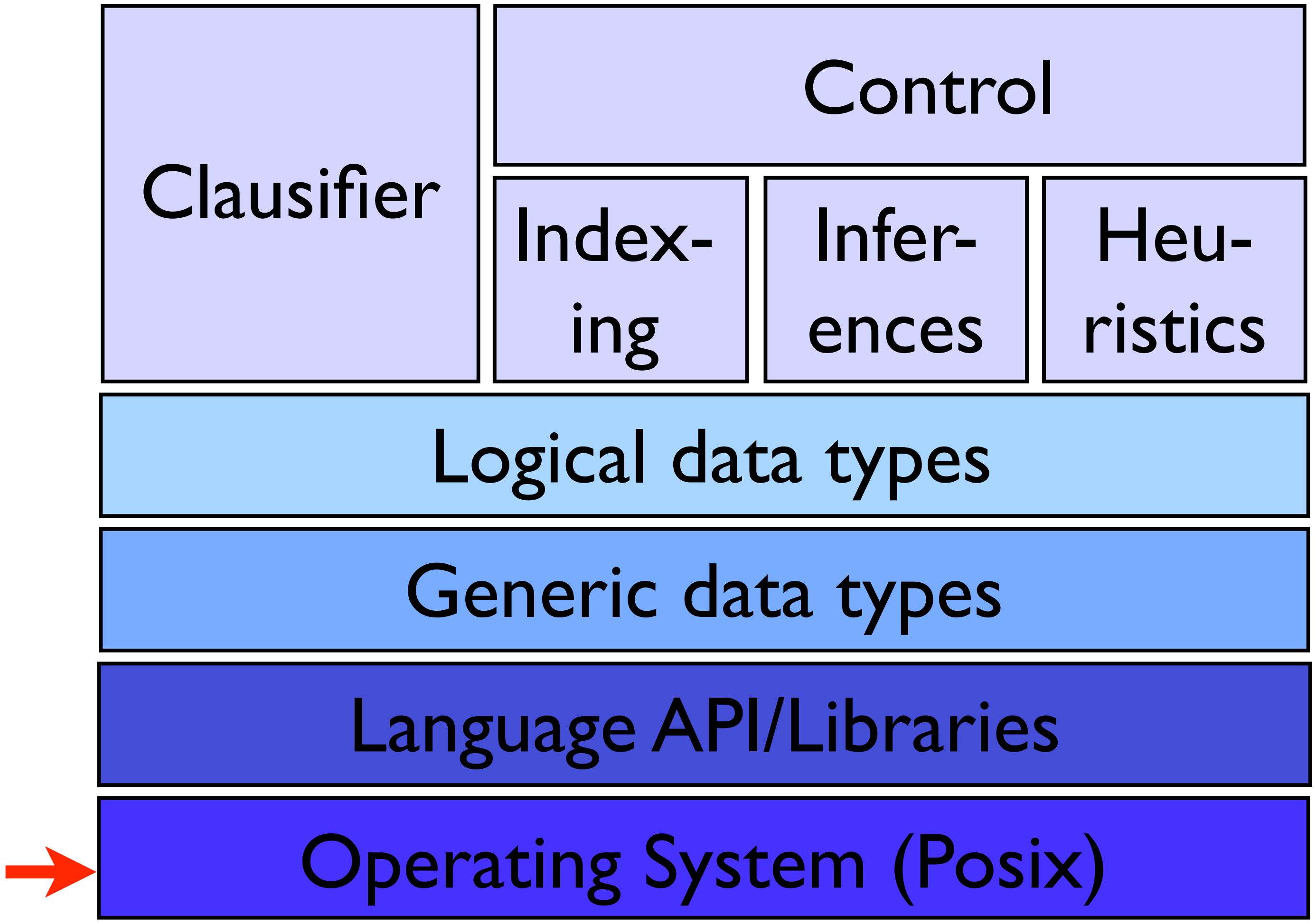
Exercise: Installing and Running E

- Goto `http://www.eprover.org`
- Find the download section
- Find and read the README
- Download the source tarball
- Following the README, build the system in a local user directory
- Run the prover on one of the included examples to demonstrates that it works.

Layered Architecture



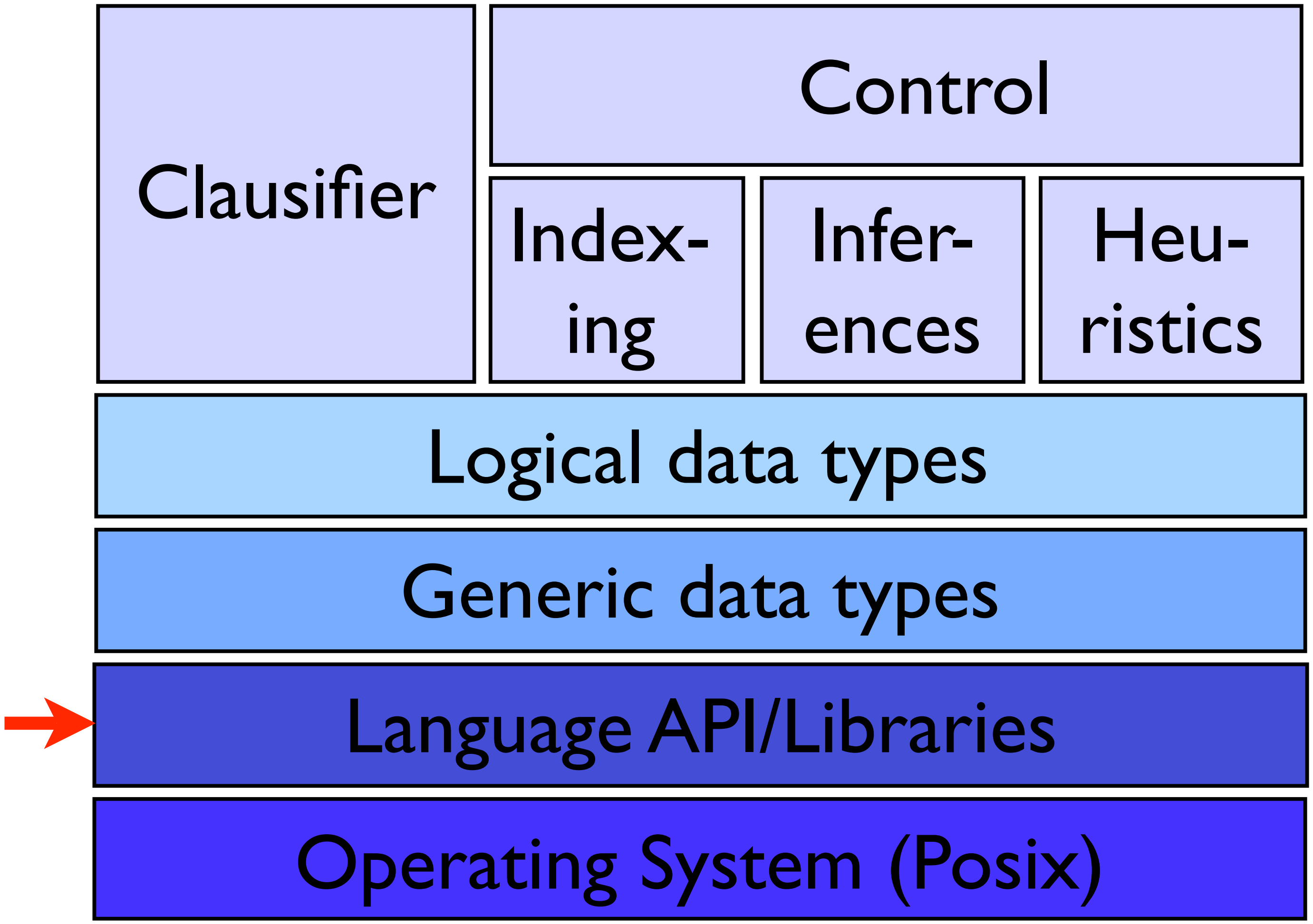
Layered Architecture



Operating System

- Pick a UNIX variant
  - Widely used
  - Free
  - Stable
  - Much better support for remote tests and automation
  - Everybody else uses it ;-)
- Aim for portability
  - Theorem provers have minimal requirements
  - Text input/output
  - POSIX is sufficient

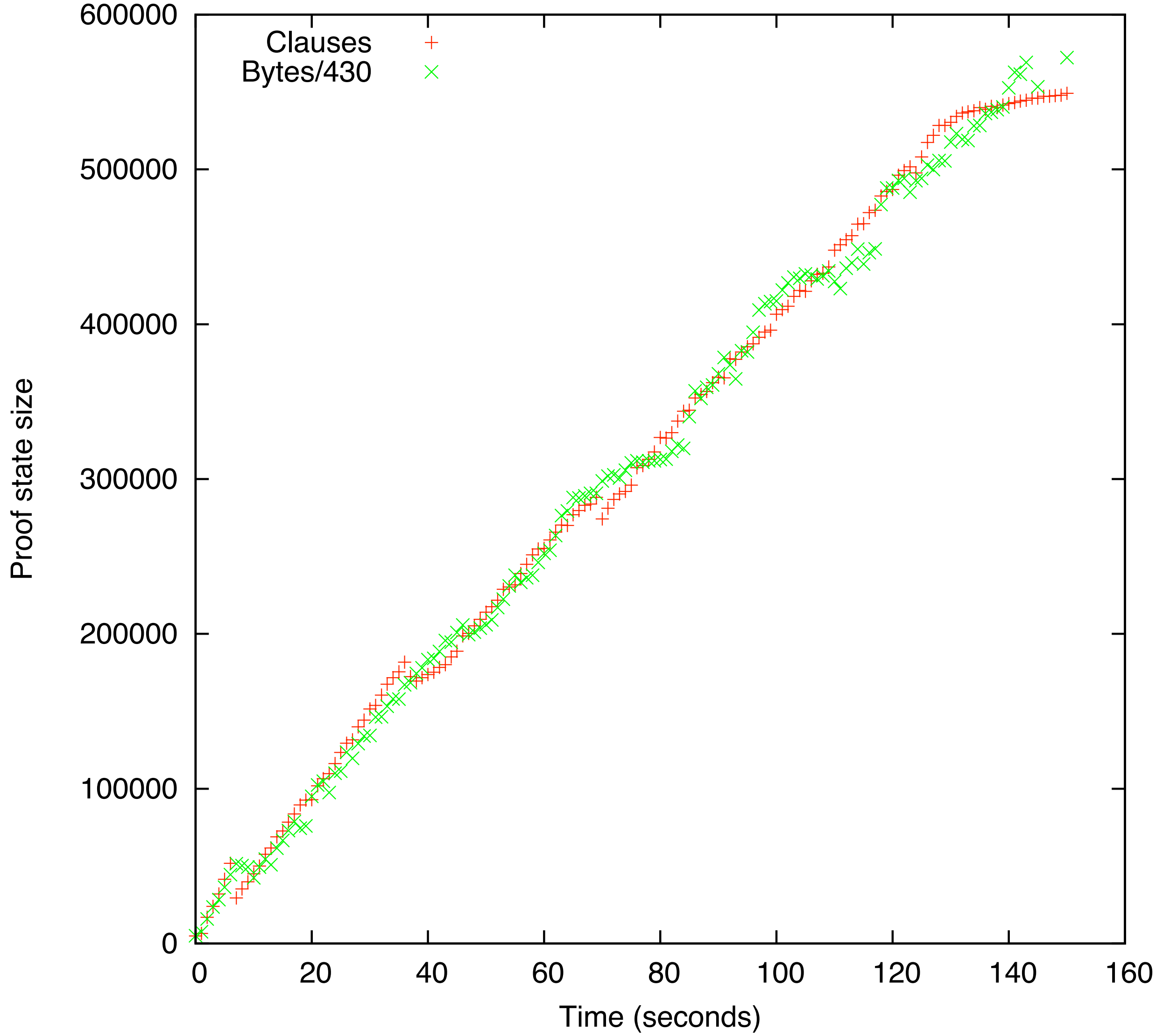
Layered Architecture



Language API/Libraries

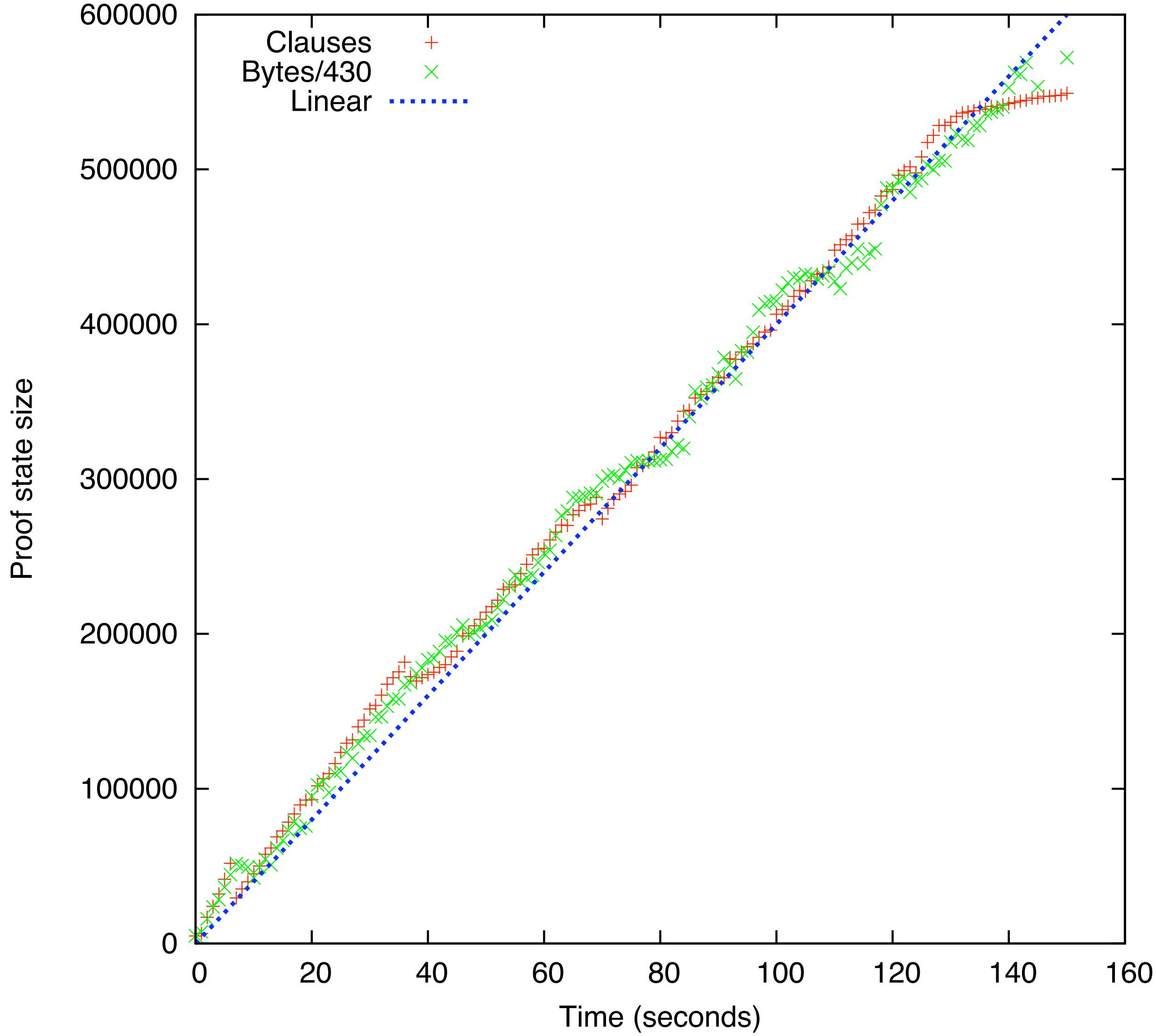
- Pick your language
- High-level/funtional or declarative languages come with rich datatypes and libraries
  - Can cover "Generic data types"
  - Can even cover 90% of "Logical data types"
- C offers nearly full control
  - Much better for low-level performance
  - . . . if you can make it happen!

Memory Consumption



- Proof state behavior for number theory example NUM030-1 (880 MHz SunFire)

Memory Consumption

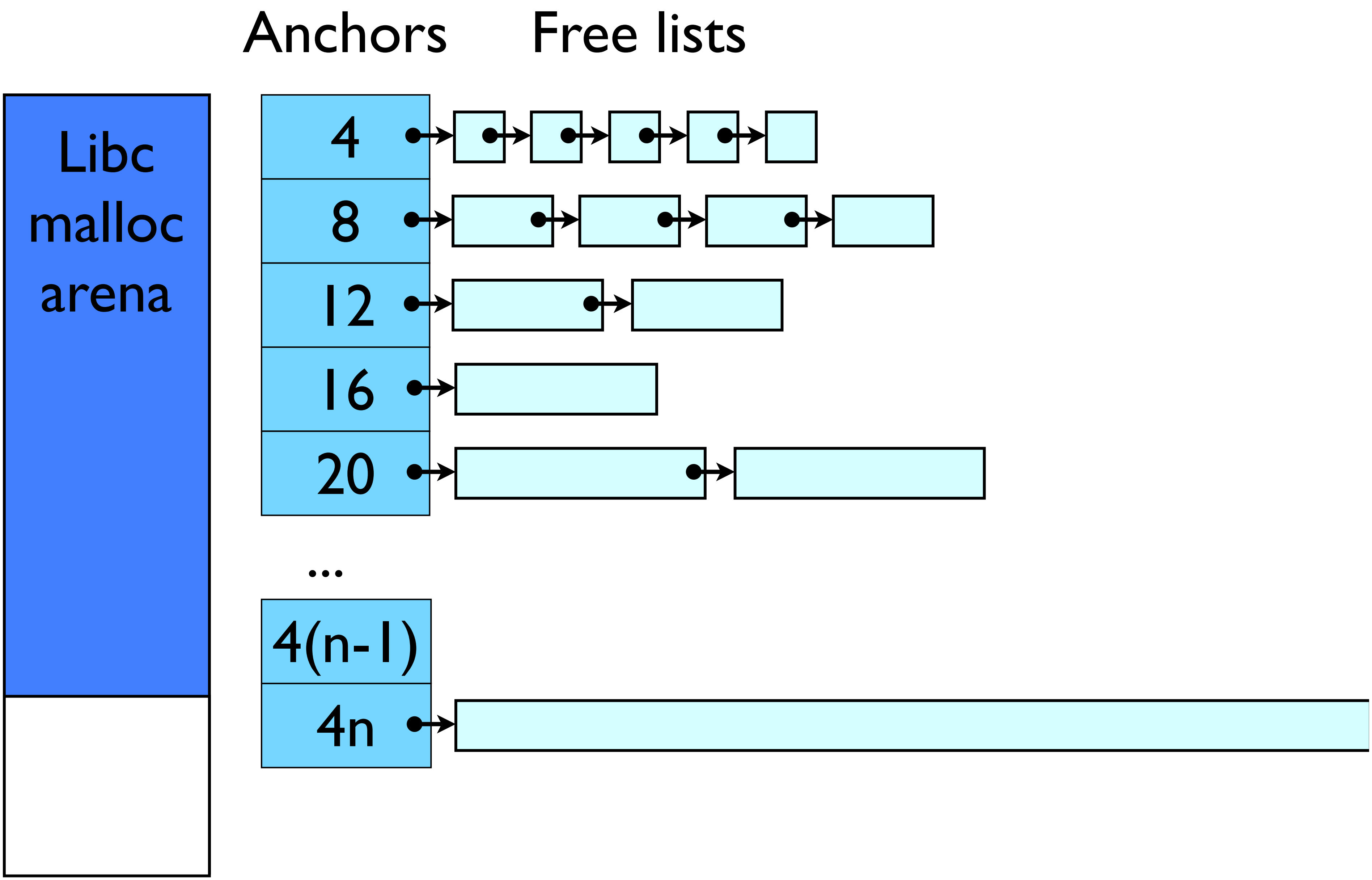


- Proof state behavior for number theory example NUM030-1 (880 MHz SunFire)

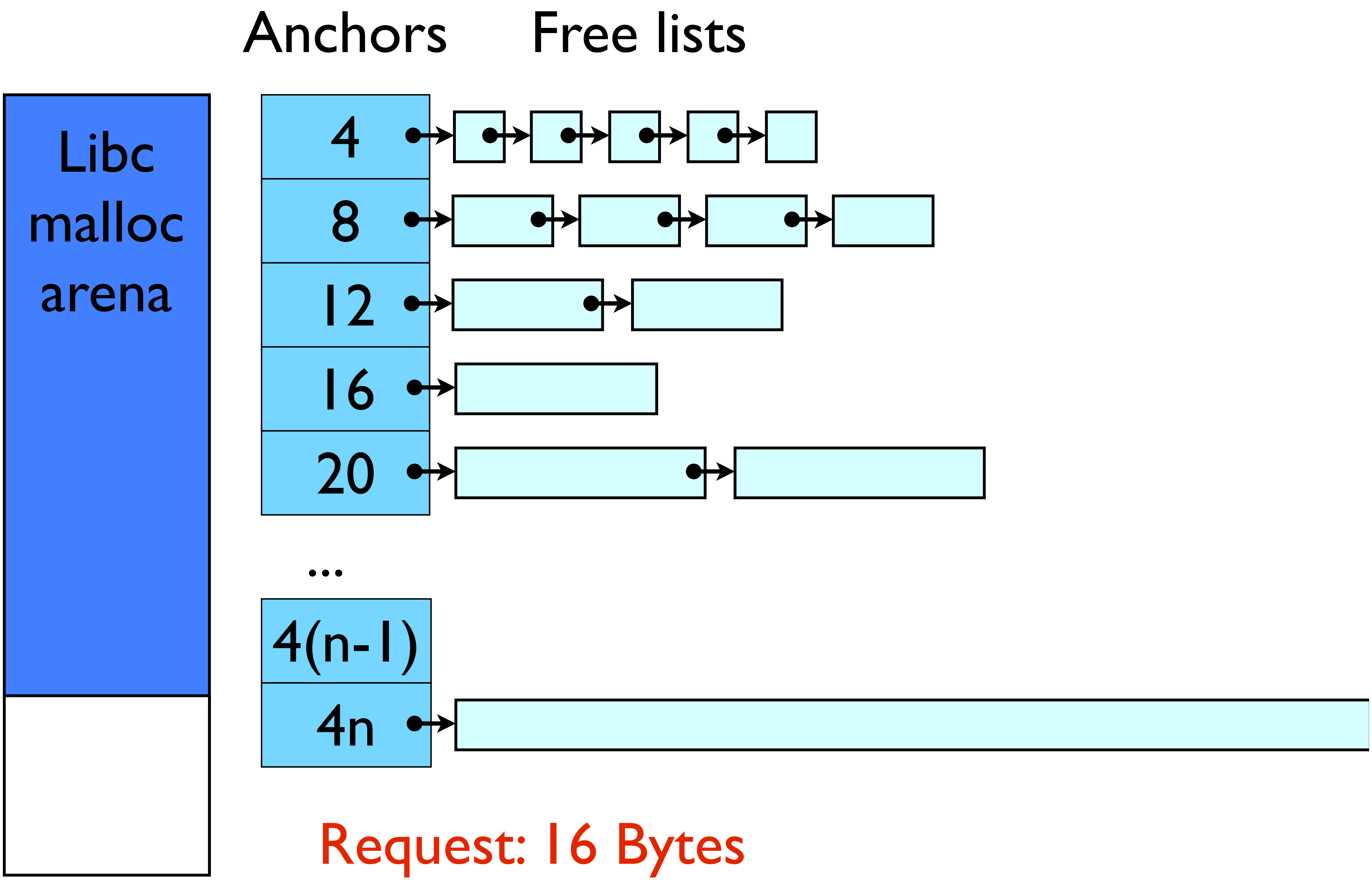
Memory Management

- Nearly all memory in a saturating prover is taken up by very few data types
  - Terms
  - Literals
  - Clauses
  - Clause evaluations
  - (Indices)
- These data types are frequently created and destroyed
  - Prime target for freelist based memory management
  - Backed directly by system malloc()
  - Allocating and chopping up large blocks does not pay off!
- Result:
  - Allocating temporary data structures is O(1)
  - Overhead is very small
  - Speedup 20%-50% depending on OS/processor/libC version

Memory Management illustrated

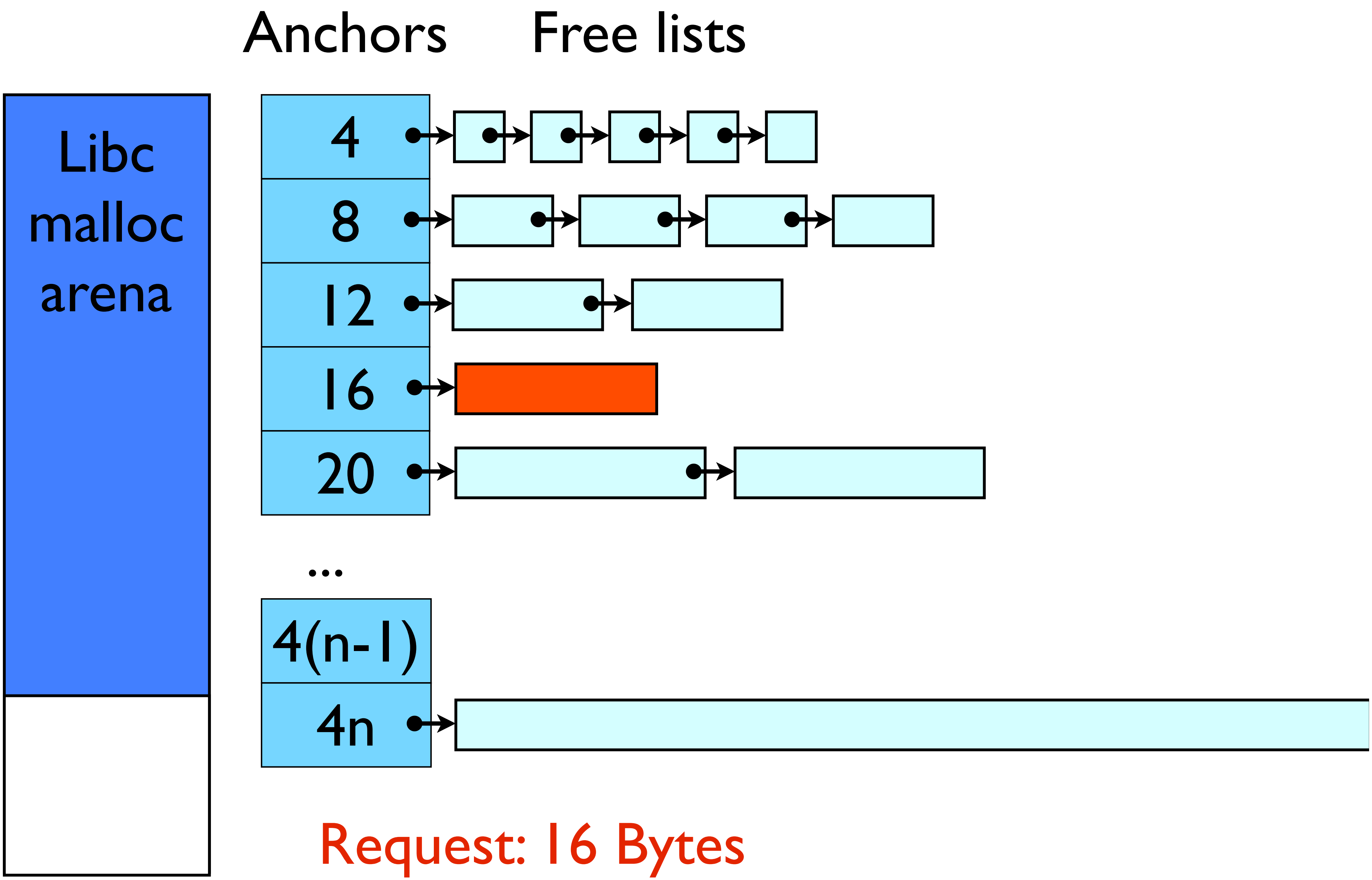


Memory Management illustrated



Request: 16 Bytes

Memory Management illustrated



Request: 16 Bytes

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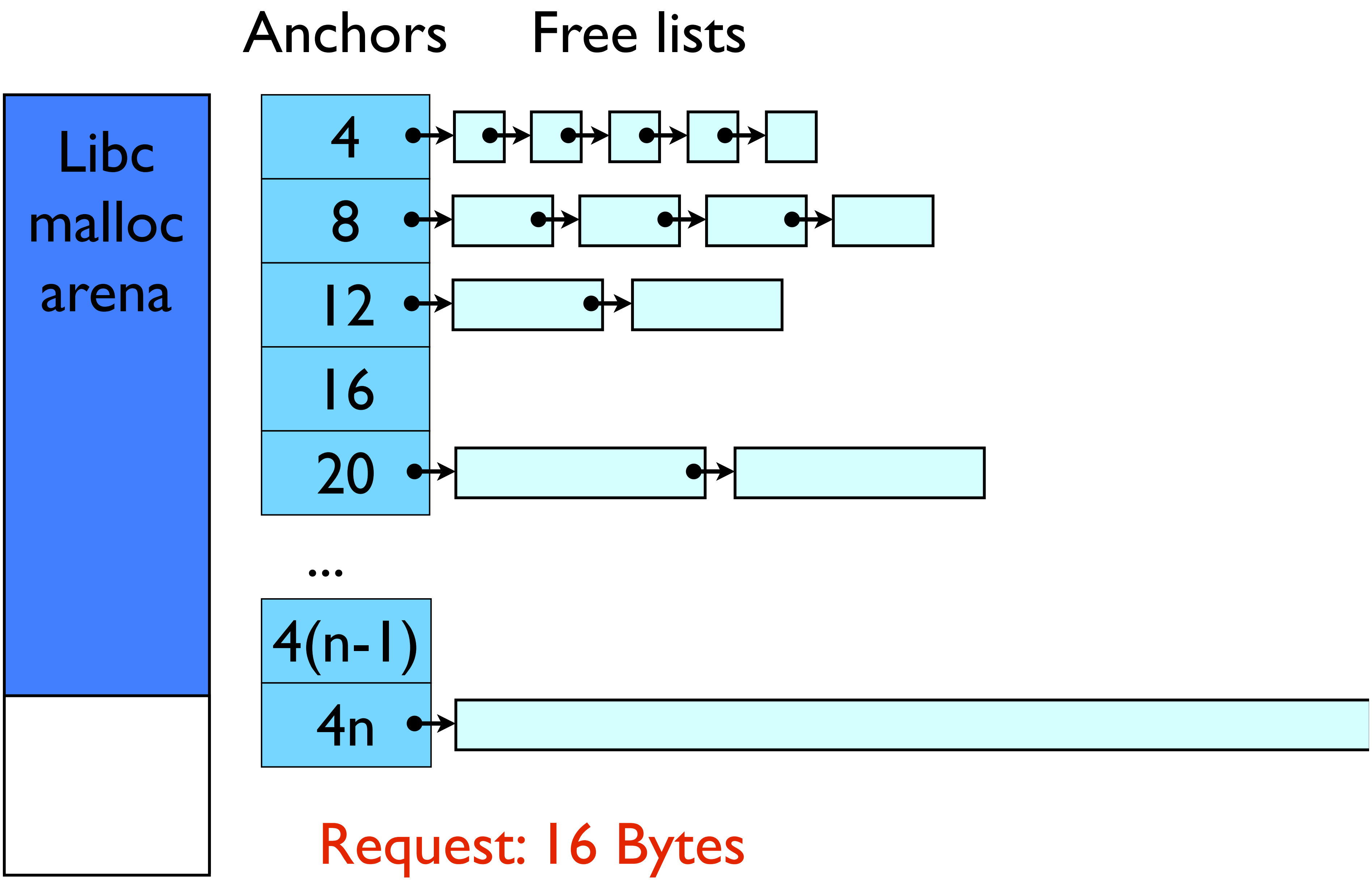

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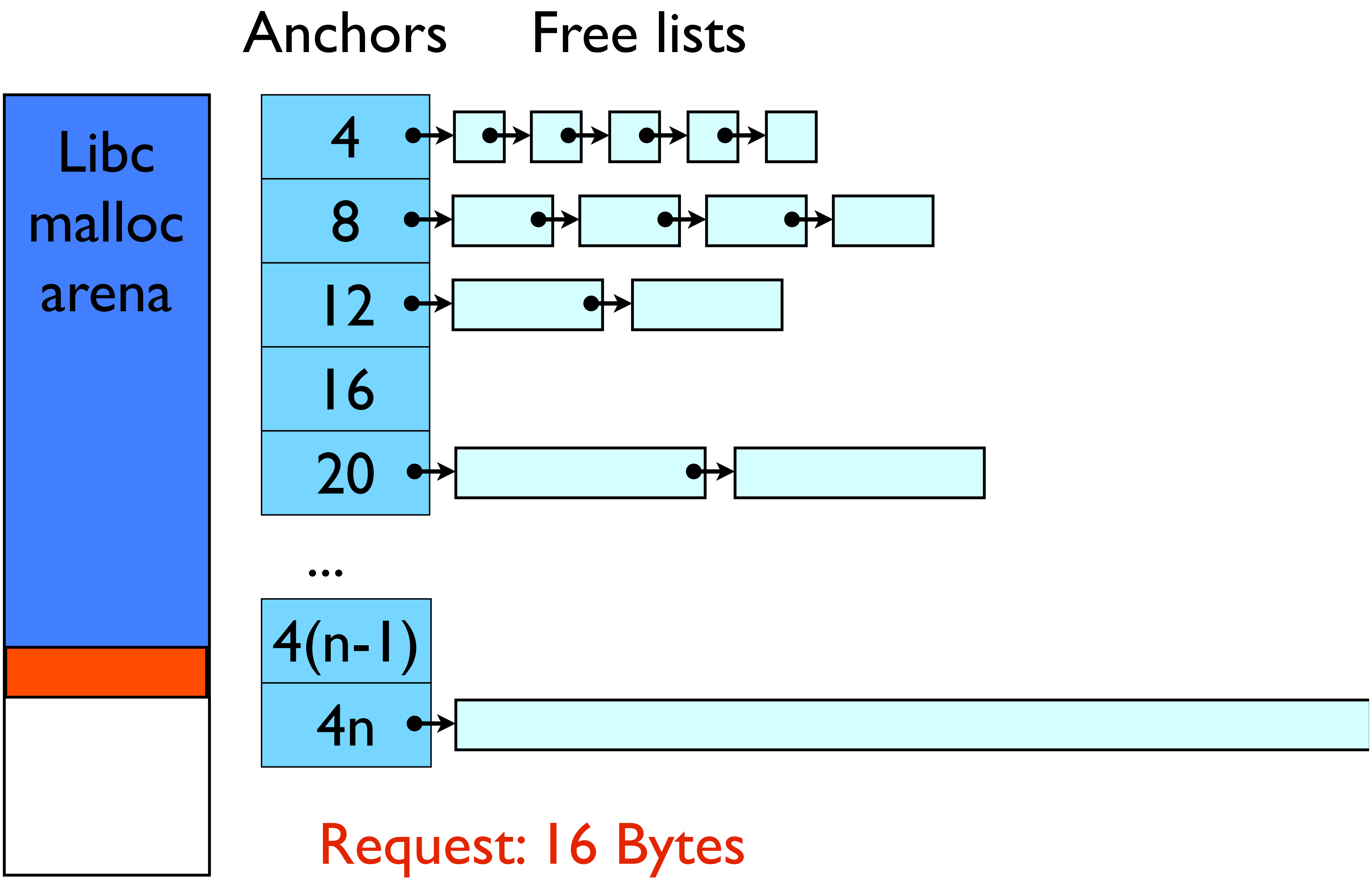
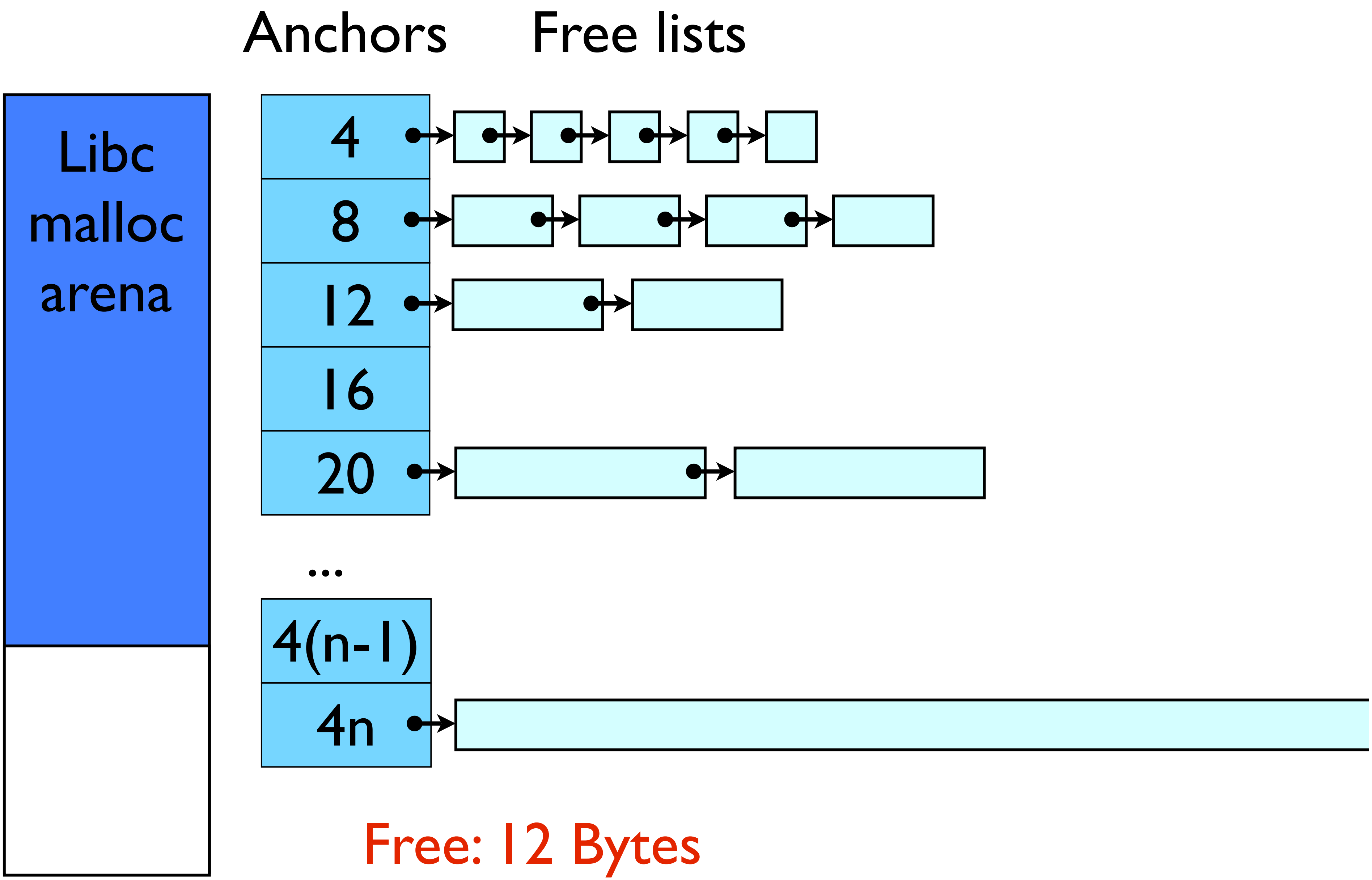
Memory Management illustrated



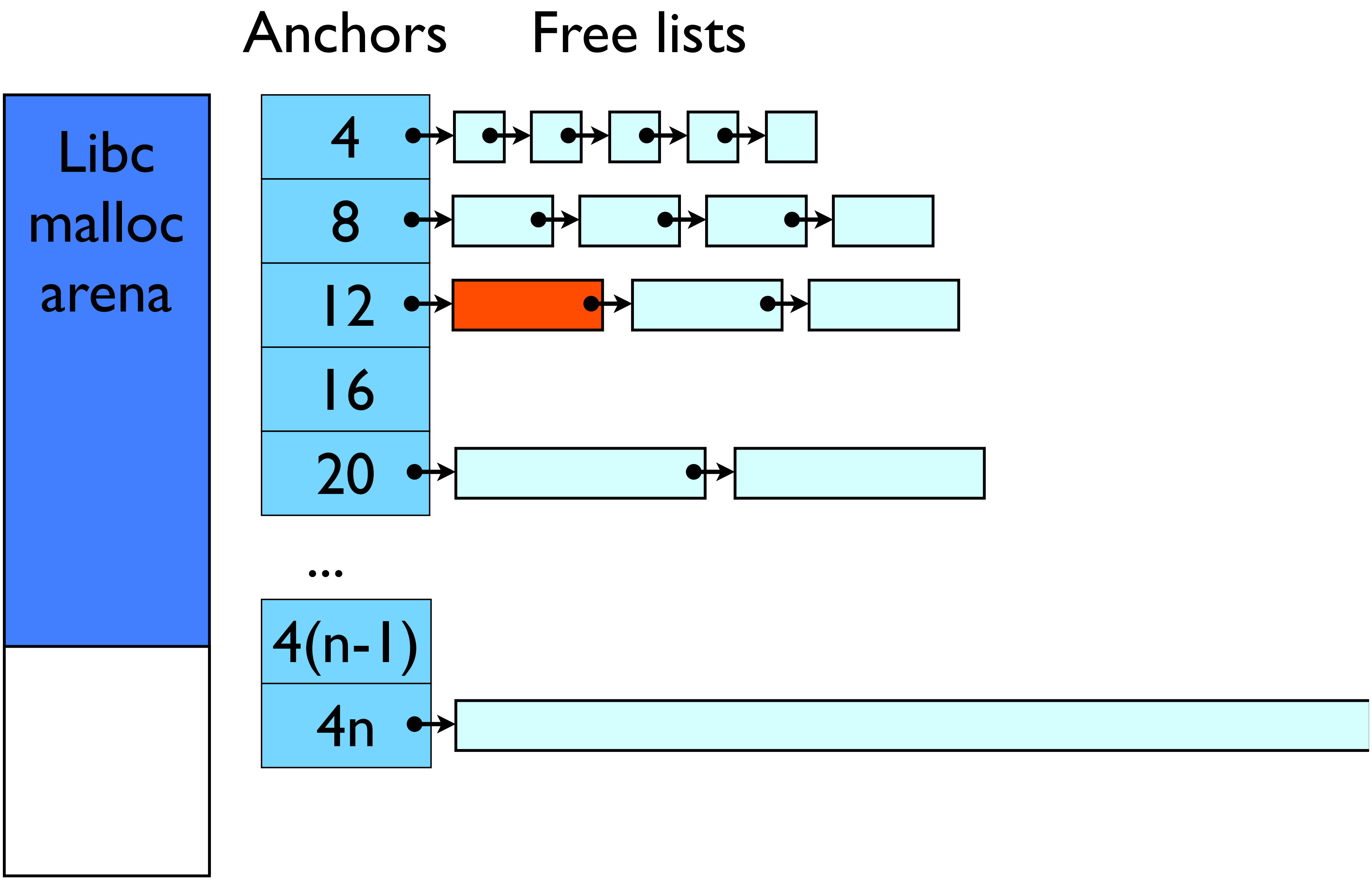
Request: 16 Bytes

Memory Management illustrated

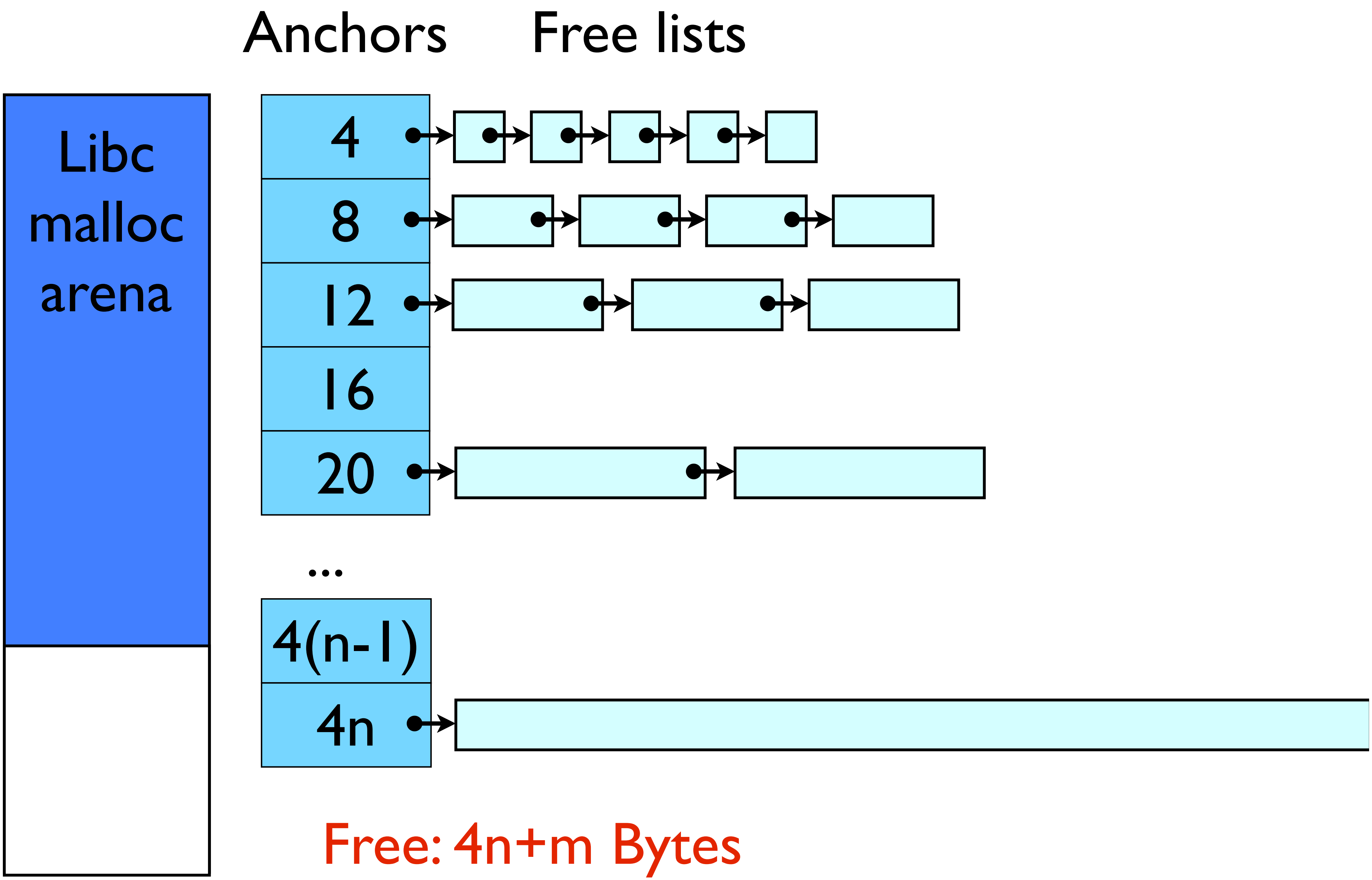


Free: 12 Bytes

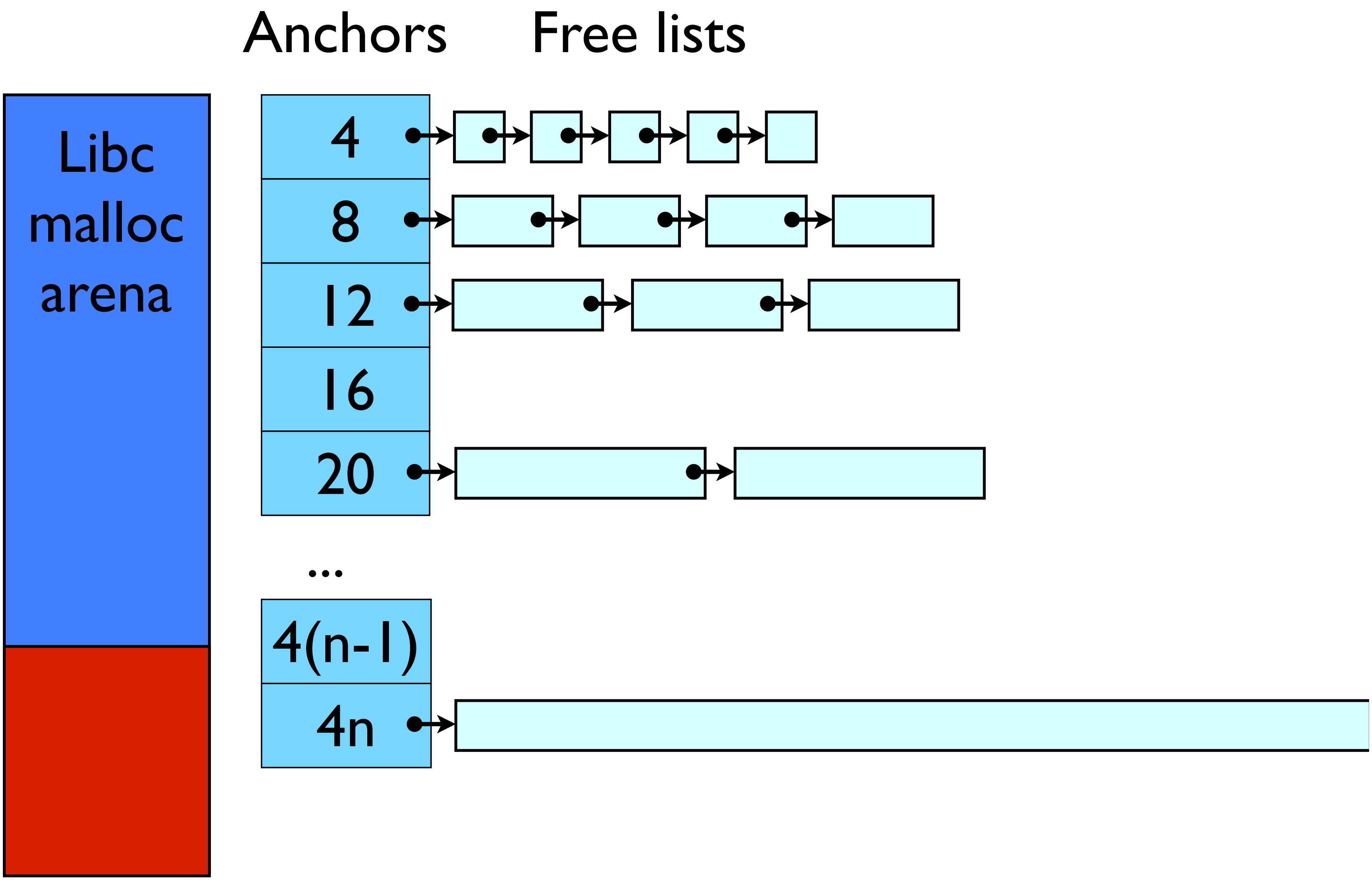
Memory Management illustrated



Memory Management illustrated



Memory Management illustrated



Exercise: Influence of Memory Management

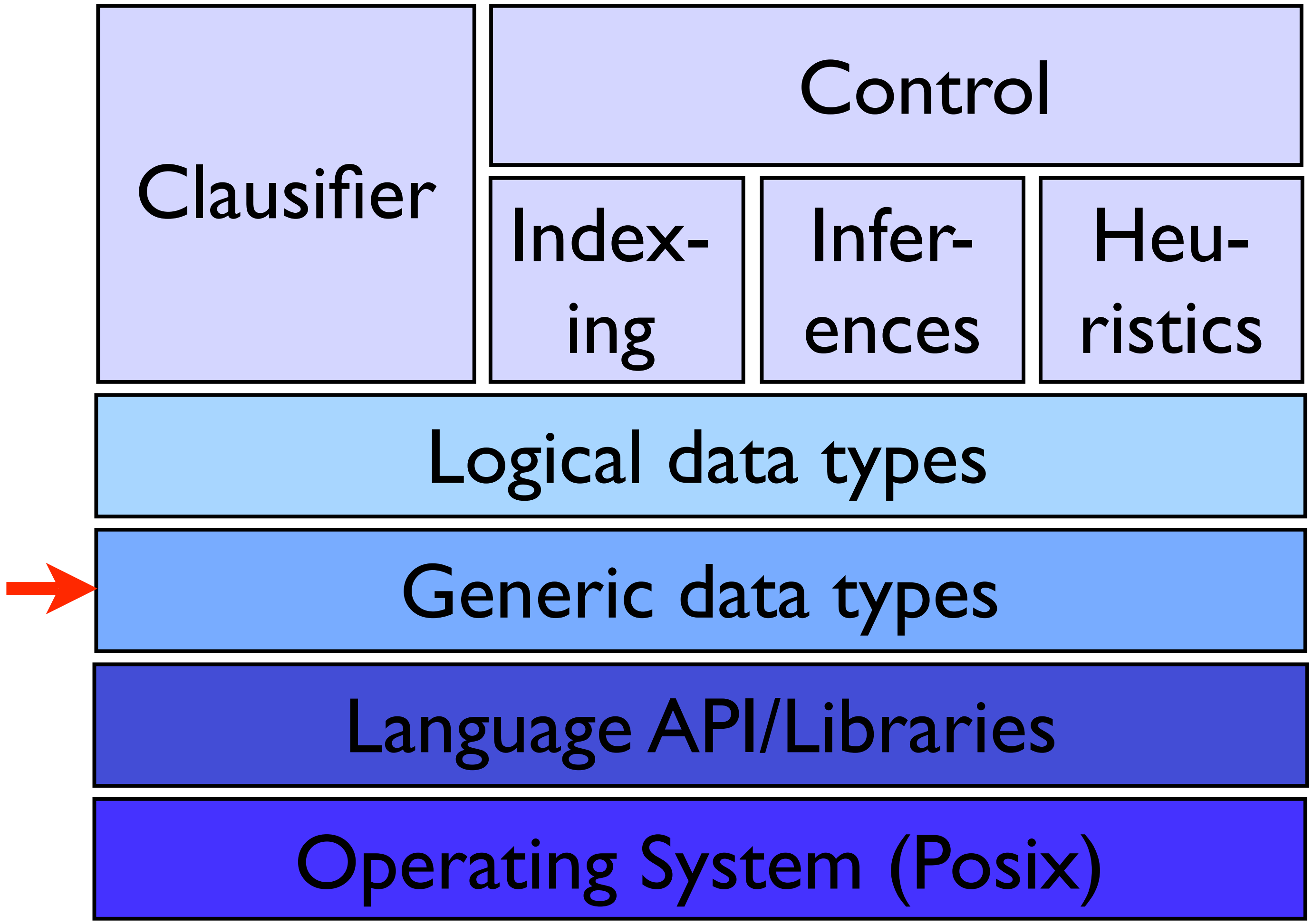
- E can be build with 2 different workin memory management schemes
  - Vanilla libC malloc()
    - Add compiler option `-DUSE_SYSTEM_MEM` in E/Makefile.vars
  - Freelists backed by malloc() (see above)
    - Default version

- Compare the performance yourself:
  - Run default E a couple of times with output disabled
  - `eprover -s --resources-info LUSK6ext.lop`
  - Take note of the reported times
  - Enable use of system malloc(), then `make rebuild`
  - Rerun the tests and compare the times

Makefile.vars

```
...
BUILDFLAGS = -DPRINT_SOMEERRORS_STDOUT \
             -DMEMORY_RESERVE_PARANOID \
             -DPRINT_TSTP_STATUS \
     -DSTACK_SIZE=32768  \
             -DUSE_SYSTEM_MEM \
             # -DFULL_MEM_STATS\
             # -DPRINT_RW_STATE # -DMEASURE_EXPENSIVE
...
```

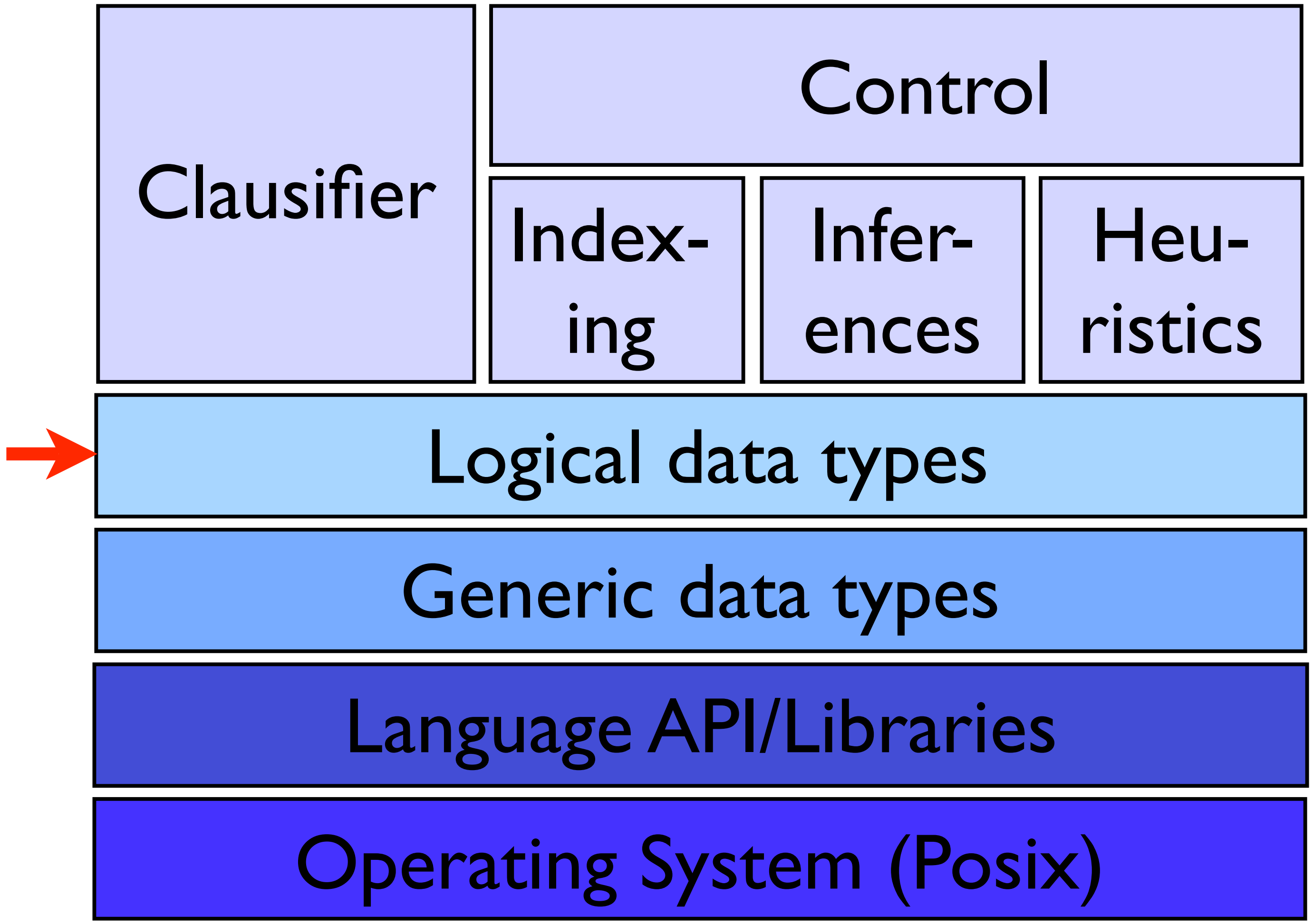
Layered Architecture



Generic Data types

- (Dynamic) Stacks
- (Dynamic) Arrays
- Hashes
- Singly linked lists
- Doubly linked lists
- Tries
- Splay trees [ST85]
- Skip lists [Pug90]

Layered Architecture



First-Order Terms

- Terms are words over the alphabet $F \cup V \cup \{'(', ')', ','\}$, where...

- Variables: $V = \{X, Y, Z, X1, \ldots\}$

- Function symbols: $F = \{f/2, g/1, a/0, b/0, \ldots\}$

- Definition of terms:

  – $X \in V$ is a term
  – $f/n \in F, t_1, \ldots, t_n$ are terms $\rightsquigarrow f(t_1, \ldots, t_n)$ is a term
  – Nothing else is a term

Terms are by far the most frequent objects in a typical proof state!
$\rightsquigarrow$ Term representation is critical!

Representing Function Symbols and Variables

- Naive: Representing function symbols as strings: `"f"`, `"g"`, `"add"`
  - May be ok for f, g, add
  - Users write $unordered\_pair, universal\_class, \ldots$
- Solution: Signature table
  - Map each function symbol to unique small positive integer
  - Represent function symbol by this integer
  - Maintain table with meta-information for function symbols indexed by assigned code
- Handling variables:
  - Rename variables to $\{X_1, X_2, \ldots\}$
  - Represent X_i by $-i$
  - Disjoint from function symbol codes!

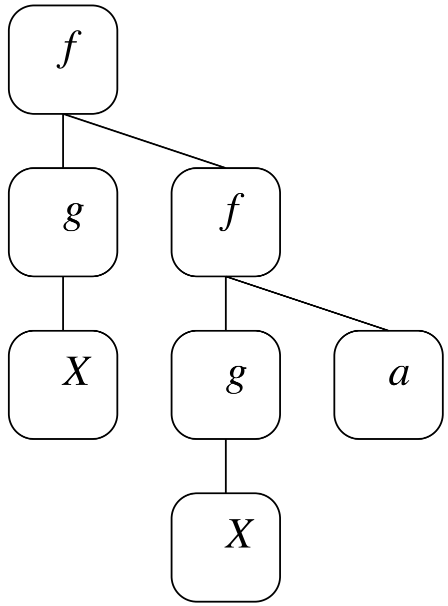
$\boxed{\text{From now on, assume this always done!}}$

Representing Terms

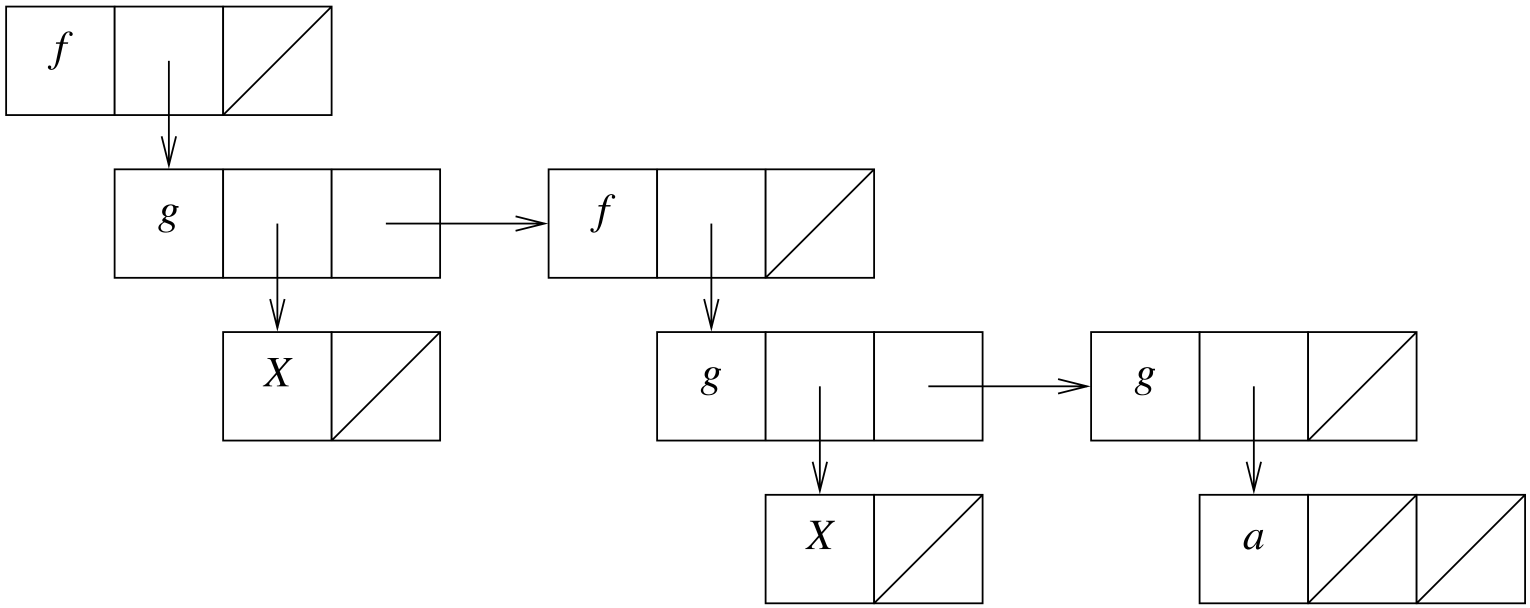
- Naive: Represent terms as strings `"f(g(X), f(g(X),a))"`
- More compact: `"fgXfgXa"`
  - Seems to be very memory-efficient!
  - But: Inconvenient for manipulation!
- Terms as ordered trees
  - Nodes are labeled with function symbols or variables
  - Successor nodes are subterms
  - Leaf nodes correspond to variables or constants
  - Obvious approach, used in many systems!

Abstract Term Trees

- Example term: $f(g(X), f(g(X), a))$

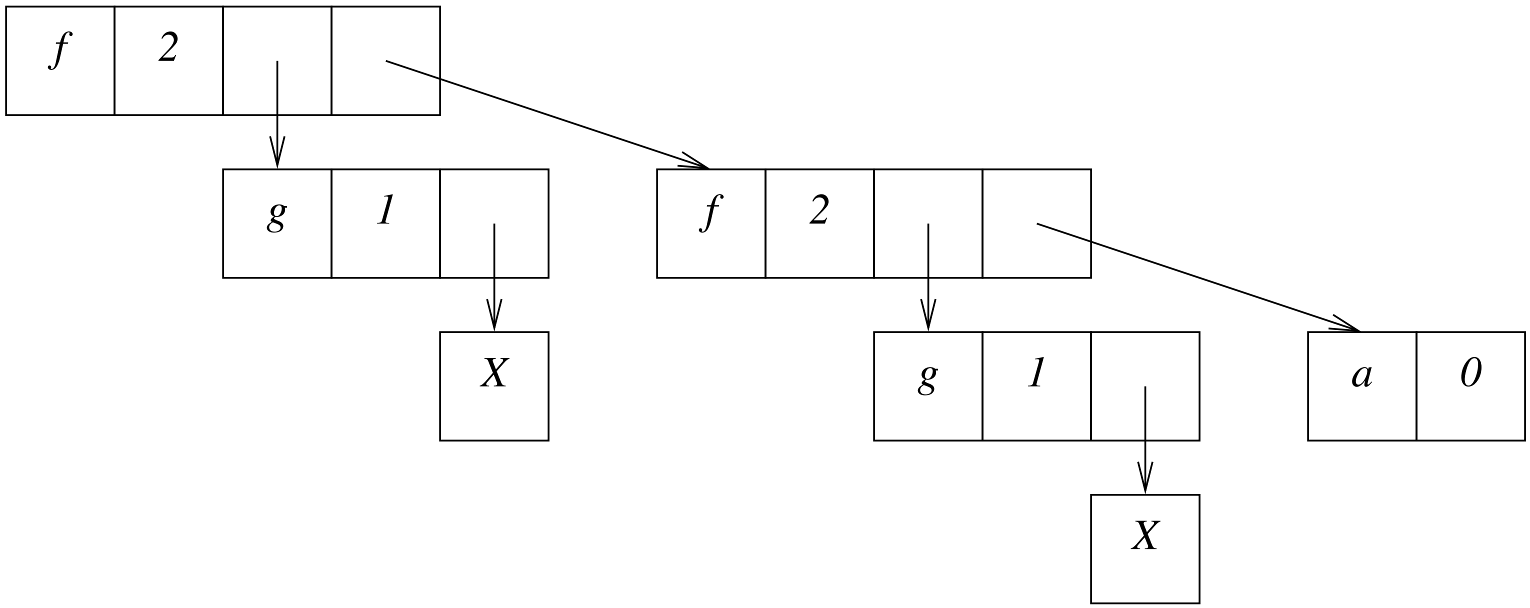


LISP-Style Term Trees



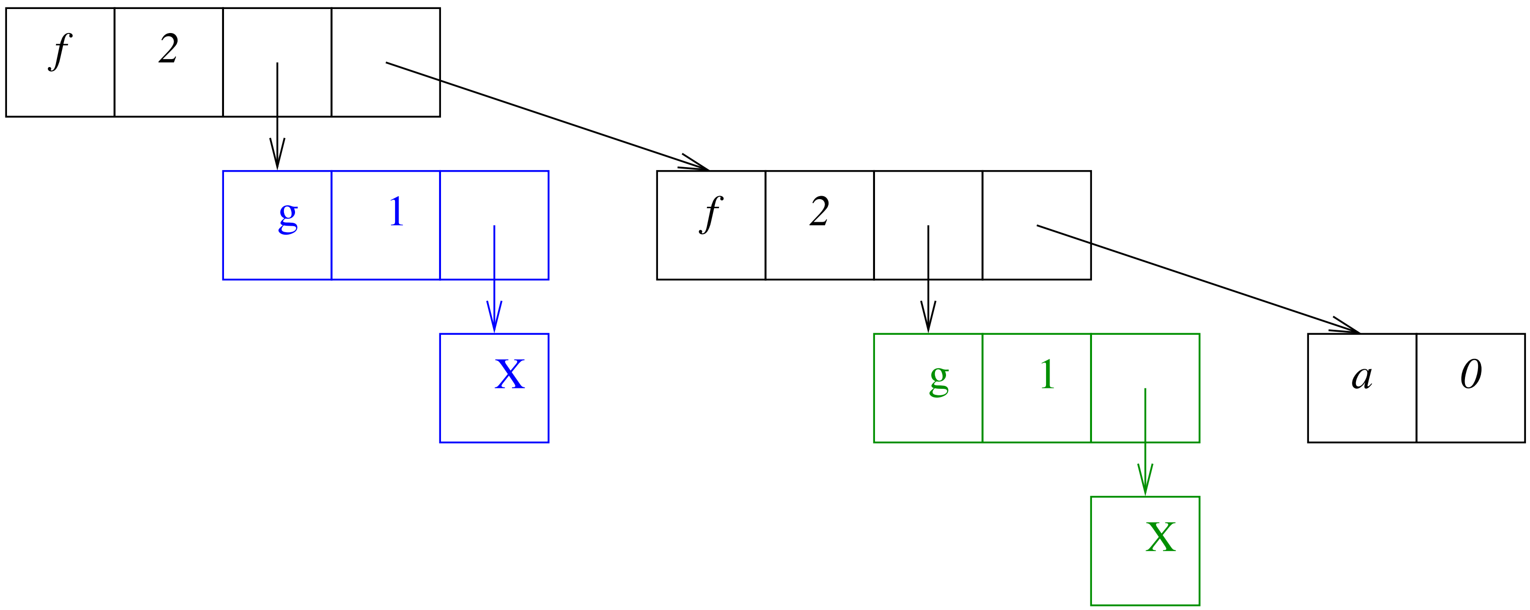
- Argument lists are represented as linked lists
- Implemented e.g. in PCL tools for DISCOUNT and Waldmeister

C/ASM Style Term Trees



- Argument lists are represented by arrays with length
- Implemented e.g. in DISCOUNT (as an evil hack)

C/ASM Style Term Trees

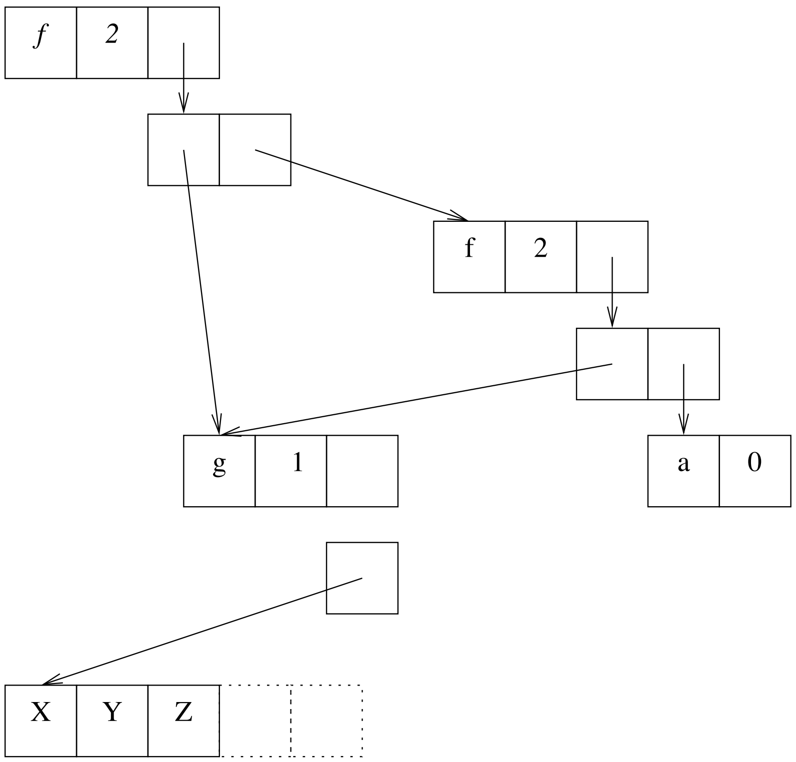


- In this version: Isomorphic subterms have isomorphic representation!

Exercise: Term Datatype in E

- E's basic term data type is defined in `E/TERMS/cte_termtypes.h`
  - Which term representation does E use?

Shared Terms (E)



- Idea: Consider terms not as trees, but as DAGs
  - Reuse identical parts
  - Shared variable banks (trivial)
  - Shared term banks maintained bottom-up

Shared Terms

- Disadvantages:
  - More complex
  - Overhead for maintaining term bank
  - Destructive changes must be avoided

- Direct Benefits:
  - Saves between 80% and 99.99% of term nodes
  - Consequence: We can afford to store precomputed values
    - Term weight
    - Rewrite status (see below)
    - Groundness flag
    - . . .
  - Term identity: One pointer comparison!

Literal Datatype

- See `E/CLAUSES/ccl_eqn.h`
- Equations are basically pairs of terms with some properties

```
/* Basic data structure for rules, equations, literals. Terms are
   always assumed to be shared and need to be manipulated while taking
   care about references! */

typedef struct eqncell
{
   EqnProperties  properties;/* Positive, maximal, equational */
   Term_p         lterm;
   Term_p         rterm;
   int            pos;
   TB_p           bank;      /* Terms are from this bank */
   struct eqncell *next;     /* For lists of equations */
}EqnCell, *Eqn_p, **EqnRef;
```

Clause Datatype

- See `E/CLAUSES/ccl_clause.h`

- Clauses are containers with Meta-information and literal lists

```
typedef struct clause_cell
{
   long                  ident;        /* Hopefully unique ident for
 all clauses created during
 proof run */
   SysDate               date;         /* ...at which this clause
 became a demodulator */
   Eqn_p                 literals;     /* List of literals */
   short                 neg_lit_no;   /* Negative literals */
   short                 pos_lit_no;   /* Positive literals */
   long                  weight;       /* ClauseStandardWeight()
 precomputed at some points in
 the program */
   Eval_p                evaluations; /* List of evauations */
```

```
   ClauseProperties      properties;  /* Anything we want to note at
 the clause? */
...
   struct clausesetcell* set;         /* Is the clause in a set? */
   struct clause_cell*   pred;        /* For clause sets = doubly  */
   struct clause_cell*   succ;        /* linked lists */
}ClauseCell, *Clause_p;
```

Summary Day 1

- First-order logic with equality
- Superposition calculus
  - Generating inferences ("Superposition rule")
  - Rewriting
  - Subsumption
- Proof procedure
  - Basic given-clause algorithm
  - DISCOUNT Loop
- Software architecture
  - Low-level components
  - Logical datetypes

Literature Online

- My papers are at `http://www4.informatik.tu-muenchen.de/~schulz/bibliography.html`
- The Workshop versions of Bernd Löchners LPO/KBO papers [Löc06, L0̈6] are published in the "Empricially Successful" series of Workshops. Proceedings are at `http://www.eprover.org/EVENTS/es_series.html`
  - "Things to know when implementing LPO": Proceedings of Empirically Successful First Order Reasoning (2004)
  - "Things to know when implementing KPO": Proceedings of Empirically Successful Classical Automated Reasoning (2005)
- Technical Report version of [BG94]:
  - `http://domino.mpi-inf.mpg.de/internet/reports.nsf/c125634c000710d4c12560410043ec01/c2de67aa270295ddc12560400038fcc3!OpenDocument`
  - . . . or Google "Bachmair Ganzinger 91-208"

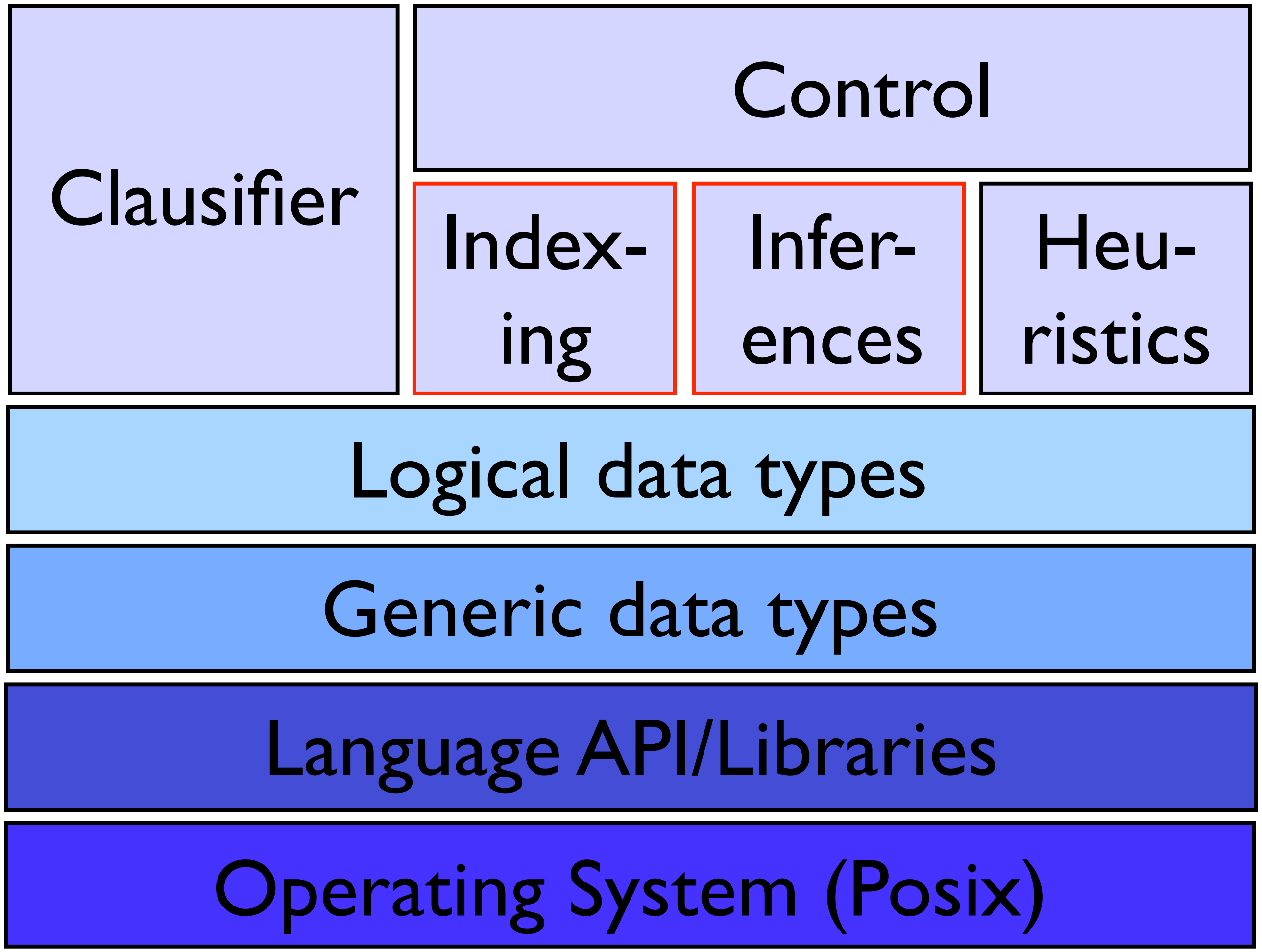
"LUSK6" Example

```
#   Problem:    In a ring, if x*x*x = x for all x
#               in the ring, then
#               x*y = y*x for all x,y in the ring.
#
#               Functions:      f   : Multiplikation *
#                               J   : Addition +
#                               g   : Inverses
#                               e   : Neutrales Element
#                               a,b : Konstanten

j (0,X)         = X.                        # 0 ist a left identity for sum
j (X,0)         = X.                        # 0 ist a right identity for sum
j (g (X),X)     = 0.                        # there exists a left inverse for sum
j (X,g (X))     = 0.                        # there exists a right inverse for sum
j (j (X,Y),Z) = j (X,j (Y,Z)).              # associativity of addition
j (X,Y)         = j(Y,X).                   # commutativity of addition
f (f (X,Y),Z) = f (X,f (Y,Z)).              # associativity of multiplication
f (X,j (Y,Z)) = j (f (X,Y),f (X,Z)).        # distributivity axioms
f (j (X,Y),Z) = j (f (X,Z),f (Y,Z)).        #
f (f(X,X),X)   = X.                         # special hypothese: x*x*x = x

f (a,b) != f (b,a).                         # (Skolemized) theorem
```

LUSK6 in TPTP-3 syntax

```
cnf(j_neutral_left,        axiom, j(0,X)          = X).
cnf(j_neutral_right,       axiom, j(X,0)          = X).
cnf(j_inverse_left,        axiom, j(g(X),X)       = 0).
cnf(j_inverse_right,       axiom, j(X,g(X))       = 0).
cnf(j_commutes,            axiom, j(X,Y)          = j(Y,X)).
cnf(j_associates,          axiom, j(j(X,Y),Z) = j(X,j(Y,Z))).
cnf(f_associates,          axiom, f(f(X,Y),Z) = f(X,f(Y,Z))).
cnf(f_distributes_left,    axiom, f(X,j(Y,Z)) = j(f(X,Y),f(X,Z))).
cnf(f_distributes_right, axiom, f(j(X,Y),Z) = j(f(X,Z),f(Y,Z))).
cnf(x_cubedequals_x,       axiom, f(f(X,X),X) = X).

fof(mult_commutes,conjecture,![X,Y]:(f(X,Y) = f(Y,X))).
```

Layered Architecture



Efficient Rewriting

- Problem:
  - Given term t, equations $E = \{l_1 \simeq r_1 \ldots l_n \simeq r_n\}$
  - Find normal form of t w.r.t. E
- Bottlenecks:
  - Find applicable equations
  - Check ordering constraint ($\sigma(l) > \sigma(r)$)
- Solutions in E:
  - Cached rewriting (normal form date, pointer)
  - Perfect discrimination tree indexing with age/size constraints

Shared Terms and Cached Rewriting

- Shared terms can be long-term persistent!
- Shared terms can afford to store more information per term node!
- Hence: Store rewrite information
  - Pointer to resulting term
  - Age of youngest equation with respect to which term is in normal form
- Terms are at most rewritten once!
- Search for matching rewrite rule can exclude old equations!

Indexing

- Quickly find inference partners in large search states
  - Replace linear search with index access
  - Especially valuable for simplifying inferences

- More concretely (or more abstractly?):
  - Given a set of terms or clauses S
  - and a query term or query clause
  - and a retrieval relation R
  - Build a data structure to efficiently find (all) terms or clauses t from S such that $R(t, S)$ (the retrieval relation holds)

Introductory Example: Text Indexing

- Problem: Given a set `D` of text documents, find all documents that contain a certain word `w`
- Obviously correct implementation:

```
result = {}
for doc in D
    for word in doc
        if w == word
            result = result ∪{ doc }
            break;
return result
```

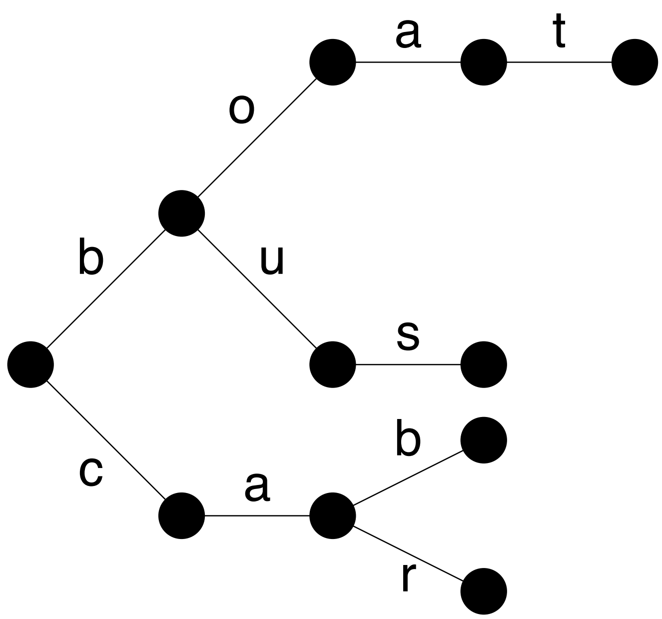
- Now think of Google. . .
  - Obvious approach (linear scan through documents) breaks down for large D
  - Instead: Precompiled Index $I : words \rightarrow documents$
  - Requirement: I efficiently computable for large number of words!

The Trie Data Structure

- Definition: Let Σ be a finite alphabet and Σ^* the set of all words over Σ
  - We write $|w|$ for the length of w
  - If $u, v \in \Sigma^*$, $w = uv$ is the word with prefix u
- A trie is a finite tree whose edges are labelled with letters from Σ
  - A node represents a set of words with a common prefix (defined by the labels on the path from the root to the node)
  - A leaf represents a single word
  - The whole trie represents the set of words at its leaves
  - Dually, for each set of words S (such that no word is the prefix of another), there is a unique trie T
- Fact: Finding the leaf representing w in T (if any) can be done in $O(|w|)$
  - This is independent of the size of S!
  - Inserting and deleting of elements is just as fast

Trie Example

- Consider $\Sigma = \{a, b, ..., z\}$ and $S = \{car, cab, bus, boat\}$
- The trie for S is:



- Tries can be built incrementally
- We can store extra infomation at nodes/leaves
  - E.g. all documents in which $boat$ occurs
  - Retrieving this information is fast and simple

Indexing Techniques for Theorem Provers

- Term Indexing standard technique for high performance theorem provers
  - Preprocess term sets into index
  - Return terms in a certain relation to a query term
    - Matches query term (find generalizations)
    - Matched by query term (find specializations)

- Perfect indexing:
  - Returns exactly the desired set of terms
  - May even return substitution

- Non-perfect indexing:
  - Returns candidates (superset of desired terms)
  - Separate test if candiate is solution

Frequent Operations

- Let S be a set of clauses
- Given term t, find an applicable rewrite rule in S
  - Forward rewriting
  - Reduced to: Given t, find $l \simeq r \in S$ such that $l\sigma = t$ for some σ
  - Find generalizations
- Given $l \rightarrow r$, find all rewritable clauses in S
  - Backward rewriting
  - Reduced to: Given l, find t such that $C|_p\sigma = l$
  - Find instances
- Given C, find a subsuming clause in S
  - Forward subsumption
  - Not easily reduced. . .
  - Backward subsumption analoguous

Classification of Indexing Techniques

- Perfect indexing
  - The index returns exactly the elements that fullfil the retrieval condition
  - Examples:
    - Perfect discrimination trees
    - Substitution trees
    - Context trees
- Non-perfect indexing:
  - The index returns a superset of the elements that fullfil the retrieval condition
  - Retrieval condition has to be verified
  - Examples:
    - (Non-perfect) discrimination trees
    - (Non-perfect) Path indexing
    - Top-symbol hashing
    - Feature vector-indexing

The Given Clause Algorithm

U: Unprocessed (passive) clauses (initially Specification)
P: Processed (active) clauses (initially: empty)

while $U \neq \{\}$
    $g =$ delete_best(U)
    $g =$ simplify(g,P)
    if $g == \square$
        SUCCESS, Proof found
    if g is not redundant w.r.t. P
        $T = \{c \in P | c$ redundant or simplifiable w.r.t. $g\}$
        $P = (P \backslash T) \cup \{g\}$
        $T = T \cup$generate(g, P)
        foreach $c \in T$
            $c =$cheap_simplify(c, P)
            if c is not trivial
                $U = U \cup \{c\}$
SUCCESS, original U is satisfiable

Typically, $|U| \sim |P|^2$ and $|U| \approx \sum |T|$

The Given Clause Algorithm

U: Unprocessed (passive) clauses (initially Specification)
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while $U \neq \{\}$
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        foreach $c \in T$
            $c =$cheap_simplify(c, P)
            if c is not trivial
                $U = U \cup \{c\}$
SUCCESS, original U is satisfiable

Simplification of new clauses is bottleneck

Sequential Search for Forward Rewriting

- Given t, find $l \simeq r \in S$ such that $l\sigma = t$ for some σ

- Naive implementation (e.g. DISCOUNT):

  function find_matching_rule(t, S)
      for $l \simeq r \in S$
          $\sigma =$ match(l, t)
          if σ and $l\sigma > r\sigma$
              return (σ, $l \simeq r$)

- Remark: We assume that for unorientable $l \simeq r$, both $l \simeq r$ and $r \simeq l$ are in S

Conventional Matching

```
match(s,t)
    return match_list([s], [t], {})

match_list(ls, lt, σ)
    while ls ≠ []
        s = head(ls)
        t = head(lt)
        if s == X ∈ V
            if X ← t' ∈ σ
                if t ≠ t' return FAIL
                    else
                        σ = σ ∪ {X ← t}
        else if t == X ∈ V return FAIL
        else
            let s = f(s_1, ..., s_n)
            let t = g(t_1, ..., t_m)
            if f ≠ g return FAIL /* Otherwise n = m! */
        ls = append(tail(ls), [s_1, ... s_n]
        lt = append(tail(lt), [t_1, ... t_m])
    return σ
```

The Size of the Problem

- Example LUSK6:
  - Run time with E on 1GHz Powerbook: 1.7 seconds
  - Final size of P: 265 clauses (processed: 1542)
  - Final size of U: 26154 clauses
  - Approximately 150,000 successful rewrite steps
  - Naive implementation: $\approx$ 50-150 times more match attempts!
  - $\approx$ 100 machine instructions/match attempt
- Hard examples:
  - Several hours on 3+GHz machines
  - Billions of rewrite attempts
- Naive implementations don't cut it!

Top Symbol Hashing

- Simple, non-perfect indexing method for (forward-) rewriting

- Idea: If $t = f(t_1, \ldots, t_n)$ ($n \geq 0$), then any s that matches t has to start with f
  - $top(t) = f$ is called the top symbol of t

- Implementation:
  - Organize $S = \cup S_f$ with $S_f = \{l \simeq r \in S | top(l) = f\}$
  - For non-variable query term t, test only rewrite rules from $S_{top(t)}$

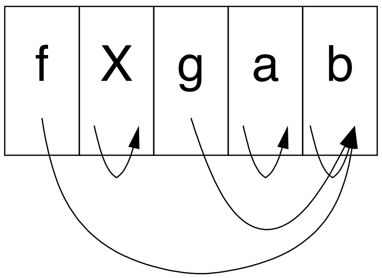
- Efficiency depends on problem composition
  - Few function symbols: Little improvement
  - Large signatures: Huge gain
  - Typically: Speed-up factor 5-15 for matching

String Terms and Flat Terms

- Terms are (conceptually) ordered trees
  - Recursive data structure
  - But: Conventional matching always does left-right traversal
  - Many other operations do likewise

- Alternative representation: String terms
  - $f(X, g(a, b))$ already is a string...
  - If arity of function symbols is fixed, we can drop braces: f$Xgab$
  - Left-right iteration is much faster (and simpler) for string terms

- Flat terms: Like string terms, but with term end pointers



  - Allows fast jumping over subterms for matching

Perfect discrimination tree indexing

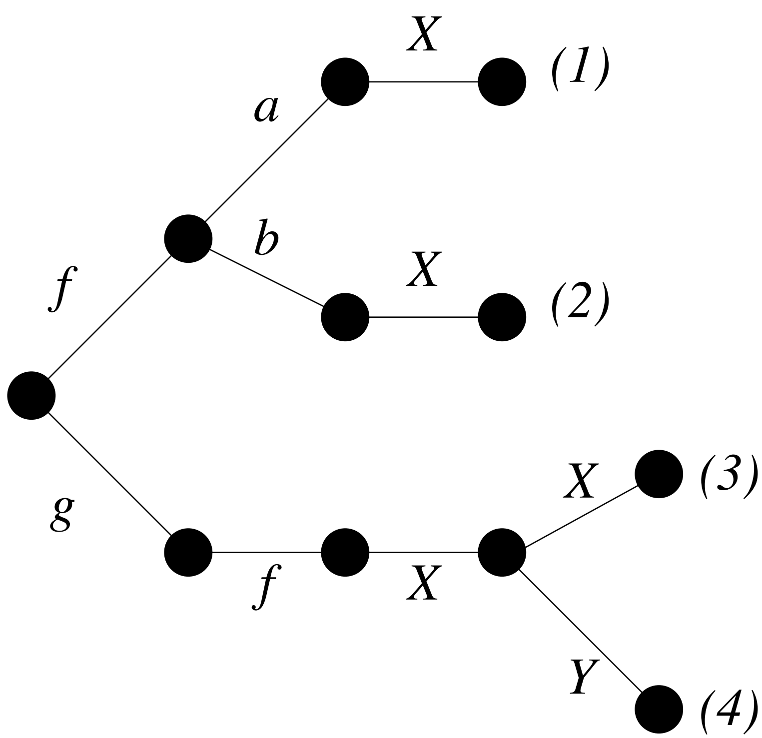
- Generalization of top symbol hashing

- Idea: Share common prefixes of terms in string representation
  - Represent terms as strings
  - Store string terms (left hand sides of rules) in trie (perfect discrimination tree)
  - Recursively traverse trie to find matching terms for a query:
    - At each node, follow all compatible vertices in turn
    - If following a variable branch, add binding for variable
    - If no valid possibility, backtrack to last open choice point
    - If leaf is reached, report match

- Currently most frequently used indexing technique
  - E (rewriting, unit subsumption)
  - Vampire (rewriting, unit- and non-unit subsumption (as code trees))
  - Waldmeister (rewriting, unit subsumption, paramodulation)
  - Gandalf (rewriting, subsumption)
  - ...

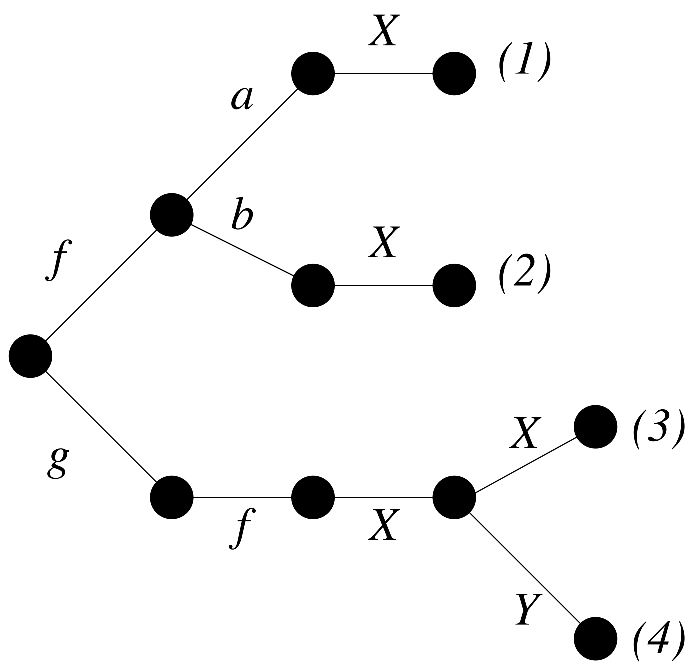
Example

- Consider $S = \{(1)f(a,X) \simeq a, (2)f(b,X) \simeq X,$
$(3)g(f(X,X)) \simeq f(Y,X), (4)g(f(X,Y)) \simeq g(X)\}$
  - String representation of left hand sides: $faX, fbX, gfXX, gfXY$
  - Corresponding trie:



Find matching rule for $g(f(a,g(b)))$

Example Continued



- Start with $g(f(a,g(b)))$, root node, $\sigma = \{\}$

  $\color{red}{g}(f(a,g(b)))$ Follow g vertex
  $g(\color{red}{f}(a,g(b)))$ Follow f vertex
  $g(f(\color{red}{a},g(b)))$ Follow X vertex, $\sigma = \{X \leftarrow a\}$, jump over a
  $g(f(a,\color{red}{g}(b)))$

  – Follow X vertex - Conflict! X already bound to a
  – Follow Y, $\sigma = \{X \leftarrow a, Y \leftarrow g(b)\}$, jump over $g(b)$ Rule 4 matches

Subsumption Indexing

- Subsumption: Important simplification technique for first-order reasoning
  - Drop less general (redundant) clauses
  - Keep more general clause
- Problem: Efficiently detecting subsumed clauses
  - Individual clause-clause subsumption is in NP
  - Large number of subsumption relations must be tested
- Major Approach: Indexing
  - Use precompiled data structures to efficiently select
    - subsuming clauses (forward subsumption)
    - subsumes clause (backward subsumption)

    from large (and fairly static) clause sets
- Usual: Different and complex indexing approaches for forward- and backward subsumption

Subsumption

- Idea: Only keep the most general clauses
  - If one clause is subsumed by another, discard it
- Formally: A clause C subsumes C' if:
  - There exists a substitution σ such that $C\sigma \subseteq C'$
  - Note: In that case $C \models C'$
  - $\subseteq$ usually is the multi-subset relation
- Examples:
  - $p(X)$ subsumes $p(a) \vee q(f(X), a)$ ($\sigma = \{X \leftarrow a\}$)
  - $p(X) \vee p(Y)$ does not multi-set-subsume $p(a) \vee q(f(X), a)$
  - $q(X, Y) \vee q(X, a)$ subsumes $q(a, a) \vee q(a, b)$
- Subsumption is hard (NP-complete)
  - $n!$ permutations in non-equational clause with n literals
  - $n!2^n$ permutations in equational clause with n literals

Forward- and Backward Subsumption

- Assume a set of clauses P and a given clause p

- Forward subsumption: Is there any clause in P that subsumes g?

- Backward subsumption: Find/remove all clauses in P subsumed by g

- Notice that these are clause–clause set operations

- Naive implementation: Sequence of clause-clause operations
  - Good implementation can speed up (average case) individual subsumption
  - Number of attempts still very high

- Smarter: Avoid many of the subsumption calls up front
  - Use indexing techniques to reduce number of candidates

Feature Vector Indexing

- New clause indexing technique
  - Not lifted from term indexing
- Advantages:
  - Small index (memory footprint)
  - Same index for forward- and backward subsumption
  - Very simple
  - Efficient in practice
  - Variants for different subsumption relations
- Disadvantages:
  - Non-perfect
  - Requires fixed signature for optimal performance

How does it work?

Properties of the Subsumption Relation

Definitions:

- Let C and C' be clauses
- C^+ is the (multi-)set (a clause) of positive literals in C
- C^- is the (multi-)set of negative literals in C
- $|C|_f$ is the number of occurences of (function or predicate) symbol f in C

Facts: If C subsumes C', then

- $|C^+| \leq |C'^+|$
- $|C^-| \leq |C'^-|$
- $|C^+|_f \leq |C'^+|_f$ for all f
- $|C^-|_f \leq |C'^-|_f$ for all f
- (Similar results exist for term depths)
- The same holds for all linear combination of these features

Remark: Composite critera are often used to detect subsumption failure early

- $|C| \leq |C'|$ (C cannot have more literals than C')
- $\sum_{f \in F} |C|_f \leq \sum_{f \in F} |C'|$ (C cannot have more symbols than C')

Feature Vectors

Definitions:

- A feature function f is a function from the set of clauses to $\mathbf{N}$
- f is subsumption-compatible, if C subsumes C' implies $f(C) \leq f(C')$
- A (subsumption-compatible) feature vector function F is a function from the set of clauses to $\mathbf{N}^n$ such that $\Pi_n^i \circ F$ (the projection of F to the ith component) is a subsumption-compatible feature function
- If v_1 and v_2 are feature vectors, we write $v_1 \leq_s v_2$, if $v_1[i] \leq v_2[i]$ for all i.

Fact:

- Assume F is a (subsumption-compatible) feature vector function
- Assume C subsumes C'
- By construction, $F(C) \leq_s F(C')$

Basic Principle of Feature Vector Indexing:

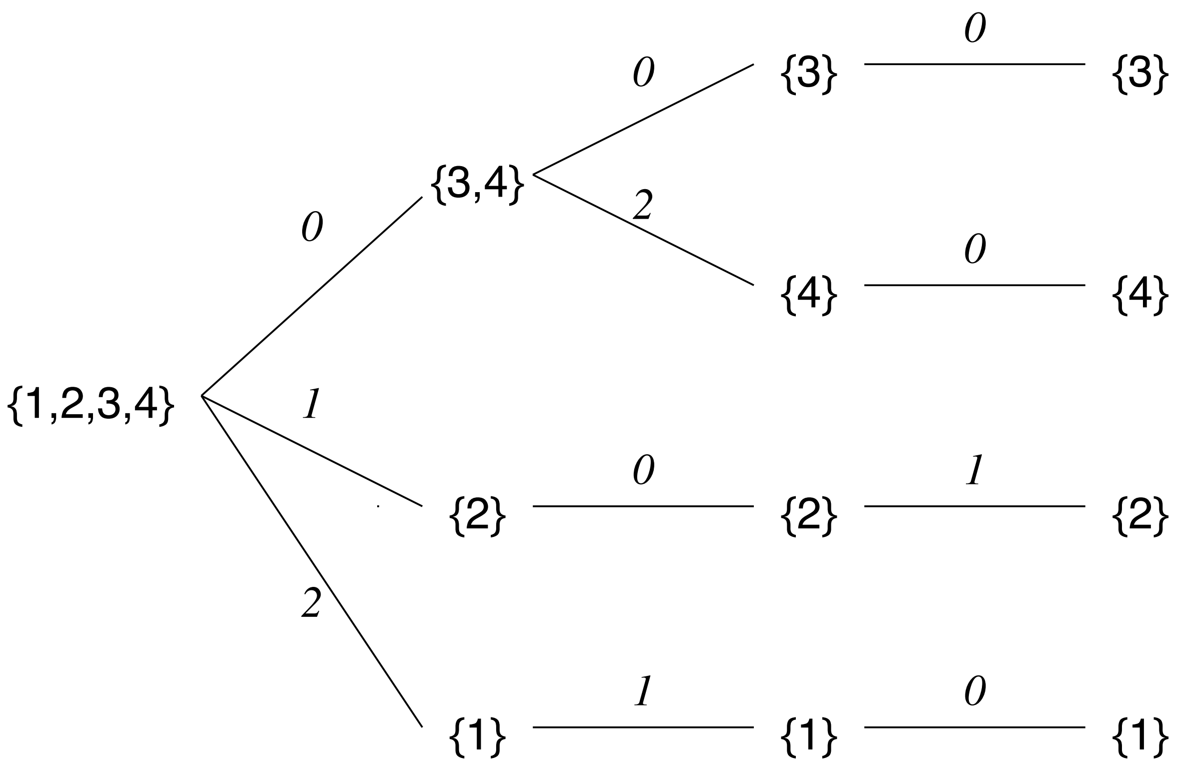
- For forward-subsumption: $candFS_F(P, g) = \{c \in P | F(c) \leq_s F(g)\}$
- For backward-subsumption: $candBS_F(P, g) = \{c \in P | F(g) \leq_s F(c)\}$

Feature Vector Indexing

- **Aim:** Efficiently compute $candFS_F(P, g)$ and $candBS_F(P, g)$
- **Solution:** Frequency vectors for P are compiled into a trie, clauses are stored in leaves
  - Tree of depth n (number of features in vector)
  - Nodes at depth d split according to feature $F(C)[d]$ (one successor per value)
  - All vectors with value $F(C)[d] = k$ associated with corresponding subtree
  - Construction continues recursively
- **Example:** Assume $F(C) := \langle |C^+|_a, |C^+|_f, |C^-|_b| \rangle$
  - Clause set $P = \{$1,2,3,4 $\}$ with
    1. $F(p(a) \vee p(f(a))) = \langle 2, 1, 0 \rangle$
    2. $F(p(a) \vee \neg p(b)) = \langle 1, 0, 1 \rangle$
    3. $F(\neg p(a) \vee p(b)) = \langle 0, 0, 0 \rangle$
    4. $F(p(X) \vee p(f(f(b)))) = \langle 0, 2, 0 \rangle$
  - Query $g = p(f(a))$
    * $F(g) = \langle 1, 1, 0 \rangle$

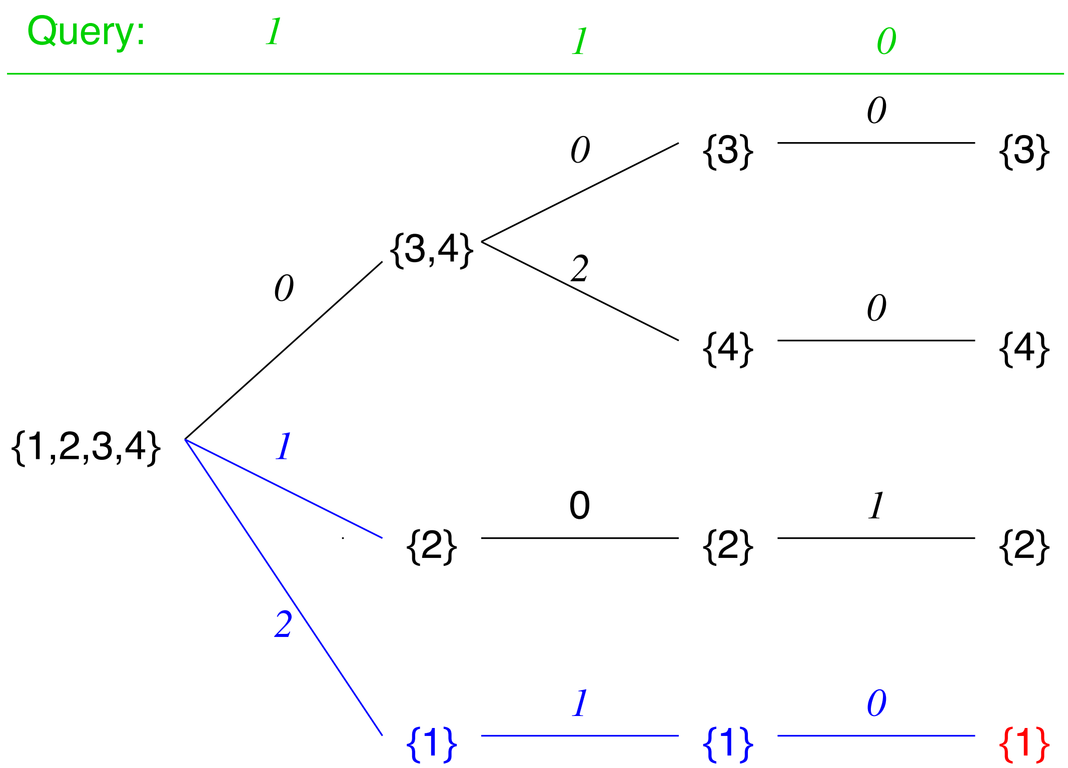
Example Index

1. $F(p(a) \vee p(f(a))) = \langle 2, 1, 0 \rangle$
2. $F(p(a) \vee \neg p(b)) = \langle 1, 0, 1 \rangle$
3. $F(\neg p(a) \vee p(b)) = \langle 0, 0, 0 \rangle$
4. $F(p(X) \vee p(f(f(b)))) = \langle 0, 2, 0 \rangle$



Example: Backward Subsumption

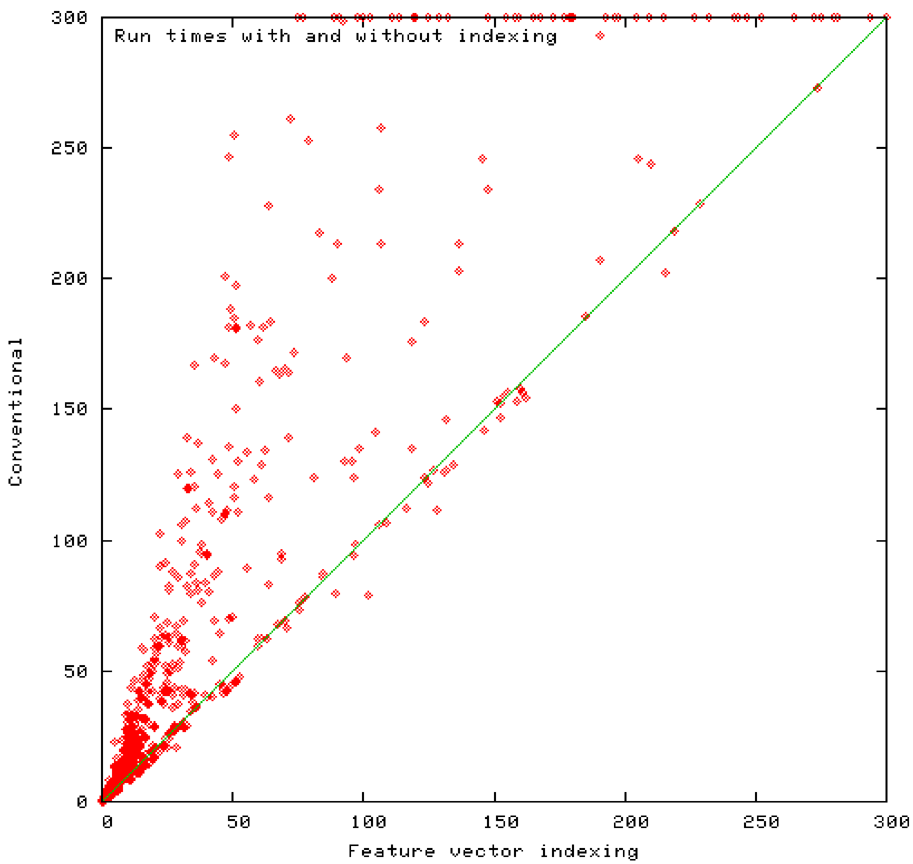
- Algorithm: At each node, only follow branches with larger or equal feature values



- Result: Just one subsumption candidate for $p(f(a))$

Performance 1

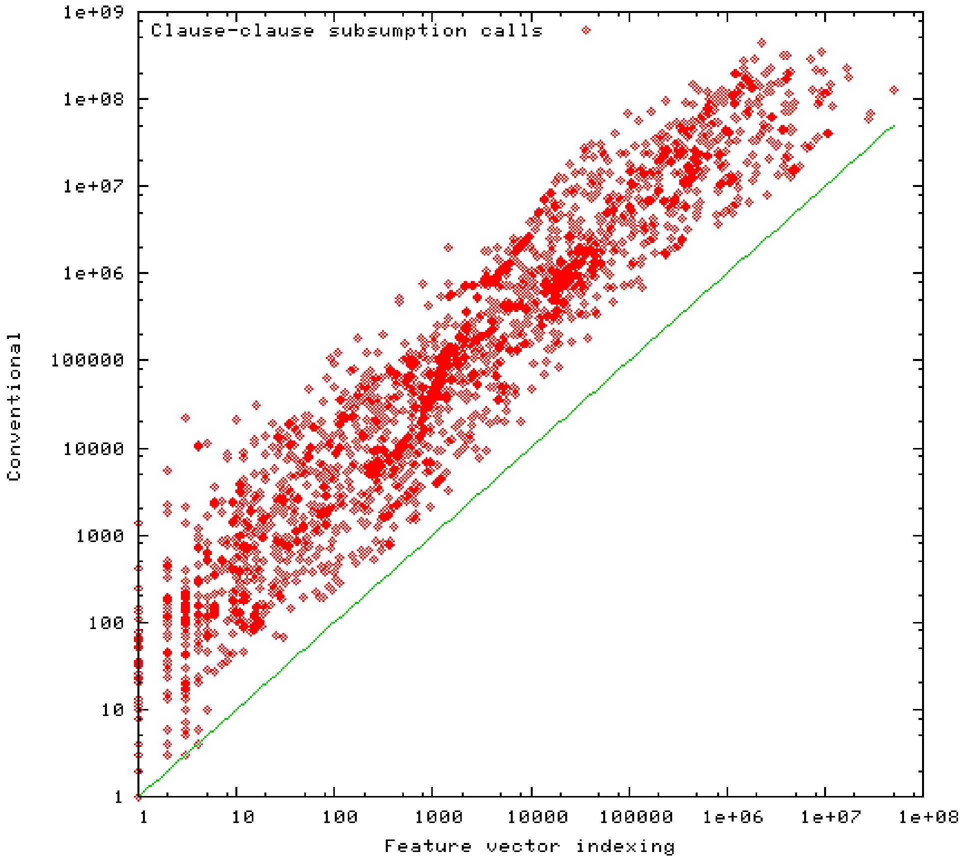
- Tested on 5180 examples from TPTP 2.5.1
  - Subsumption-heavy search strategy (contextual literal cutting)
  - Max. 75 features, 300MHz SUN Ultra 60, 300s time limit



- Speedup ca. 40%, overhead usually insignificant, 2717 vs. 2671 solutions found

Performance 2

- Number of subsumption attempts (notice double log scale)



- Average reduction: $1:60$, max: $1:8000 (1:\infty)$

Literature on Indexing

- Overview: [Gra95, SRV01]

- Classic paper: [McC92]

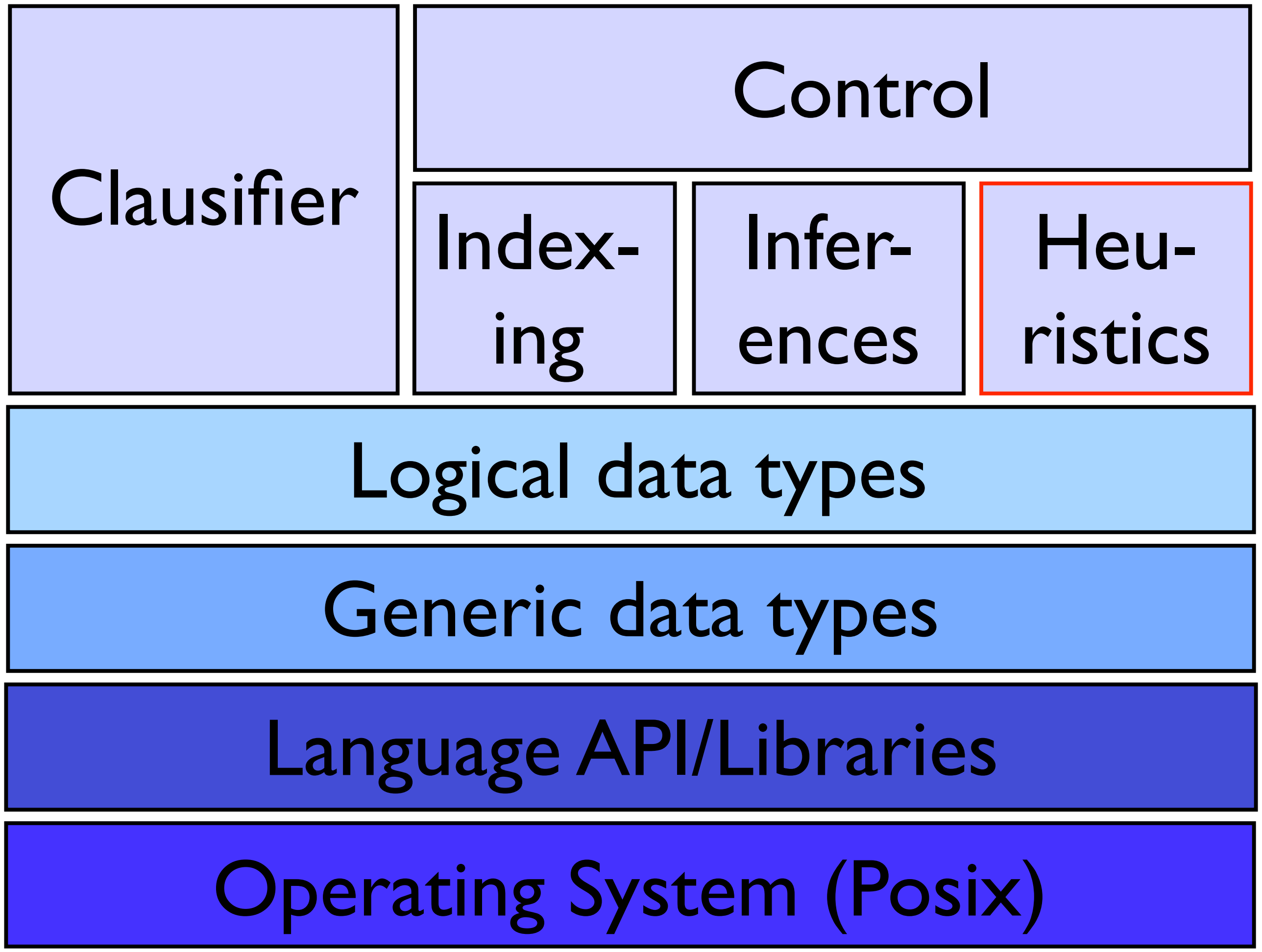
- Comparisons (for rewriting): [NHRV01]

- Feature vector indexing: [Sch04a]

Excercise: Unification

- E's unification code is SubstComputeMgu() in `E/TERMS/cte_match_mgu_1-1.[hc]`
  - Read and understand the code
  - Unification is broken down into subtassk
  - Subtasks are stored in a particular order
  - Why? Experiment with different orders!

Layered Architecture



Don't-care-Nondeterminism $\equiv$ Chances for Heuristics

- Important choice points for E:
  - Simplification ordering
  - Clause selection
  - Literal selection

- Other choice points:
  - Choice of rewrite relation (usually strongest, don't care which normal form)
  - Application of rewrite relation to terms (leftmost-innermost, strongly suggested by shared terms)

Simplification Orderings

- Implemented: Knuth-Bendix-Orderings, Lexicographic Path Orderings
- Precedence: Fully user defined or simple algorithms
  - **Sorted by arity** (higher arity $\rightarrow$ larger)
  - **Sorted by arity, but unary first**
  - Sorted by inverse arity
  - Sorted by frequency of appearance in axioms
  - ...
- Weights for KBO: Similar simple algorithms (constant weights (optionally weight 0 for maximal symbol), arity, position in precedence ...)
- No good automatic selection of orderings yet – auto mode switches between two simple KBO schemes

Clause selection

- Most important choice point (?)
- Probably also hardest chocice (find best clause among millions)
- Implementation in E: Multiple priority queues sorted by heuristic evaluation and strategy-defined priority
- Selection in weighted round-robin-scheme (generalizes pick-given ratio)
- Example: 8*Refinedweight(PreferGoals,1,2,2,3,0.8),
  8*Refinedweight(PreferNonGoals,2,1,2,3,0.8),
  1*Clauseweight(ConstPrio,1,1,0.7),
  1*FIFOWeight(ByNegLitDist)
- Big win: Goal directed search
  – Symbols in the goal have low (=good) weights
  – Other symbols have increasingly large weight based on linking distance

Literal Selection

- Problem: Which literals should be selected for inferences in a clause?
- Ideas:
  - Select hard literals first (if we cannot solve this, the clause is useless)
  - Select small literals (fewer possible overlaps)
  - Select ground literals (no instantiation, most unit-overlaps eleminated by rewriting)
  - Propagate inference literals to children clauses (inheritance)
- Problem: Should we always select literals if possible?
  - Only select if no unique maximal literal exists
  - Do not select in conditional rewrite rules
- Surprisingly successful: Additional selection of maximal positive literals
- See E source code for large number of things we have tried. . .

Literature on other Systems

- Real (saturating) provers: [LH02, RV02, Sch02, Wei01, WSH+07, Sti92, Sti89, LS01b]

- Significant alternative approaches:
  - DCTP [SL01, LS01a, LS02],
  - Model elimination: SETHEO [LSBB92, MIL+97], leanCOP [OB03, Ott08]
  - Instantiation-Based Reasoning: iProver: [Kor08, Kor09]
  - Model Evolution: Darwin [BFT06]

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