

Formal Languages and Automata

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with contributions from David Suendermann

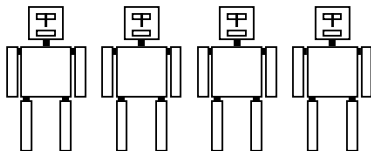


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Selected Solutions

- ▶ Stephan Schulz
 - ▶ Dipl.-Inform., U. Kaiserslautern, 1995
 - ▶ Dr. rer. nat., TU München, 2000
 - ▶ Visiting professor, U. Miami, 2002
 - ▶ Visiting professor, U. West Indies, 2005
 - ▶ Lecturer (Hildesheim, Offenburg, ...) since 2009
 - ▶ Industry experience: Building Air Traffic Control systems
 - ▶ System engineer, 2005
 - ▶ Project manager, 2007
 - ▶ Product Manager, 2013
 - ▶ Professor, DHBW Stuttgart, 2014

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Research: Logic & Automated Reasoning

- ▶ Jan Hladik
 - ▶ Dipl.-Inform.: RWTH Aachen, 2001
 - ▶ Dr. rer. nat.: TU Dresden, 2007
 - ▶ Industry experience: SAP Research
 - ▶ Work in publicly funded research projects
 - ▶ Collaboration with SAP product groups
 - ▶ Supervision of Bachelor, Master, and PhD students
 - ▶ Professor: DHBW Stuttgart, 2014

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**Research: Semantic Web, Semantic Technologies,
Automated Reasoning**

▶ Scripts

- ▶ The most up-to-date version of this document as well as auxiliary material will be made available online at

`http://wwwlehre.dhbw-stuttgart.de/
~sschulz/fla2018.html`

and

`http://wwwlehre.dhbw-stuttgart.de/
~hladik/FLA`

▶ Books

- ▶ John E. Hopcroft, Rajeev Motwani, Jeffrey D. Ullman: [Introduction to Automata Theory, Languages, and Computation](#)
- ▶ Michael Sipser: [Introduction to the Theory of Computation](#)
- ▶ Dirk W. Hoffmann: [Theoretische Informatik](#)
- ▶ Ulrich Hedtstück: [Einführung in die theoretische Informatik](#)

Computing Environment

- ▶ For practical exercises, you will need a complete Linux/UNIX environment. If you do not run one natively, there are several options:
 - ▶ You can install VirtualBox (<https://www.virtualbox.org>) and then install e.g. Ubuntu (<http://www.ubuntu.com/>) on a virtual machine
 - ▶ For Windows, you can install the **complete** UNIX emulation package Cygwin from <http://cygwin.com>
 - ▶ For MacOS, you can install `fink` (<http://fink.sourceforge.net/>) or MacPorts (<https://www.macports.org/>) and the necessary tools
- ▶ You will need at least `flex`, `bison`, `gcc`, `make`, and a good text editor

Outline of the Lecture

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Regular Languages and Finite Automata

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- Properties of Regular Languages

Scanners and Flex

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Formal language concepts

Alphabet: finite set Σ of symbols (characters)

▶ $\{a, b, c\}$

Word: finite sequence w of characters (string)

▶ $ab \neq ba$

Language: (possibly infinite) set L of words

▶ $\{ab, ba\} = \{ba, ab\}$

Formal: L defined precisely

▶ opposed to **natural** languages, where there are borderline cases

Some formal languages

Example

- ▶ names in a phone directory
- ▶ phone numbers in a phone directory
- ▶ legal C identifiers
- ▶ legal C programs
- ▶ legal [HTML 4.01 Transitional](#) documents
- ▶ empty set
- ▶ ASCII strings
- ▶ Unicode strings

Some formal languages

Example

- ▶ names in a phone directory
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- ▶ legal C identifiers
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- ▶ legal [HTML 4.01 Transitional](#) documents
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More?

Language classes

This course: four classes of different complexity and expressivity

- 1 **regular** languages: limited power, but easy to handle
 - ▶ “strings that start with a letter, followed by up to 7 letters or digits”
 - ▶ legal C identifiers
 - ▶ phone numbers

Language classes

This course: four classes of different complexity and expressivity

- 1 regular** languages: limited power, but easy to handle
 - ▶ “strings that start with a letter, followed by up to 7 letters or digits”
 - ▶ legal C identifiers
 - ▶ phone numbers
- 2 context-free** languages: more expressive, but still feasible
 - ▶ “every `<token>` is matched by `</token>`”
 - ▶ **nested** dependencies
 - ▶ (most aspects of) legal C programs
 - ▶ many natural languages (English, German)

Jan says that we
let
the children
help
Hans
paint
the house

Jan sagt, dass wir
die Kinder
dem Hans
das Haus
anstreichen
helfen
ließen

Language classes (cont')

- 3 **context-sensitive** languages: even more expressive, difficult to handle computationally
 - ▶ “every variable has to be declared before it is used”
(arbitrary sequence, arbitrary amounts of code in between)
 - ▶ **cross-serial** dependencies
 - ▶ (remaining aspects of) legal C programs
 - ▶ most remaining natural languages

Language classes (cont')

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 - ▶ most remaining natural languages (Swiss German)

```
Jan säit das mer  
d' chind  
  em Hans  
    es huus  
lönd  
  helfe  
    aastriche
```

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- 4 **recursively enumerable** languages: most general (Chomsky) class; undecidable
 - ▶ all (valid) mathematical theorems (in first-order logic)
 - ▶ programs terminating on a particular input

Automata

- ▶ abstract formal machine model, characterised by **states, letters, transitions, and external memory**
- ▶ **accept** words

Automata

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For every language class discussed in this course, a machine model exists such that for every **language** L there is an **automaton** $\mathcal{A}(L)$ that accepts exactly the words in L .

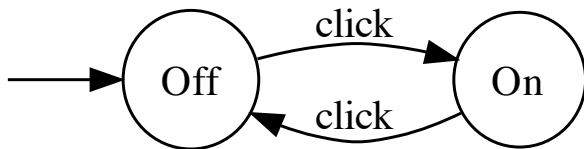
Automata

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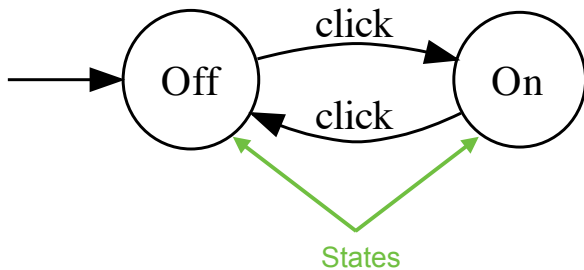
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regular	\rightsquigarrow	finite automaton/finite state machine
context-free	\rightsquigarrow	pushdown automaton
context-sensitive	\rightsquigarrow	linearly bounded Turing machine
recursively enumerable	\rightsquigarrow	(unbounded) Turing machine

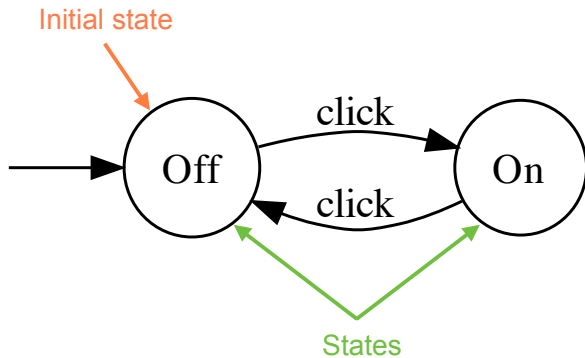
Example: Finite Automaton



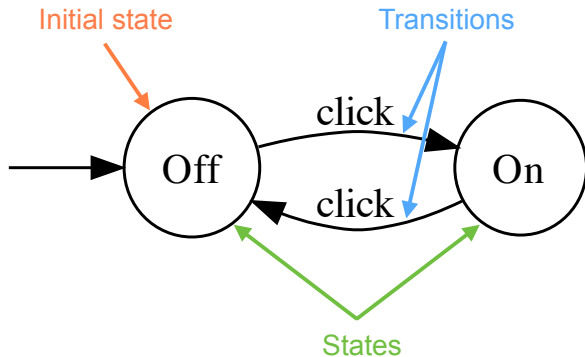
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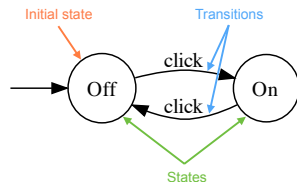
Example: Finite Automaton

Formally:

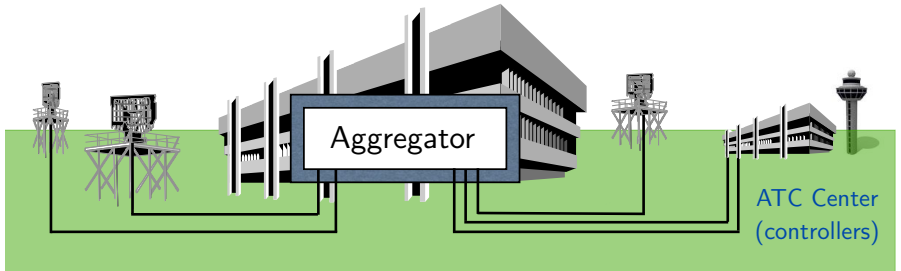
- ▶ $Q = \{\text{Off}, \text{On}\}$ is the set of **states**
- ▶ $\Sigma = \{\text{click}\}$ is the **alphabet**
- ▶ The **transition function** δ is given by

δ	click
Off	On
On	Off

- ▶ The **initial state** is Off
- ▶ There are no **accepting states**



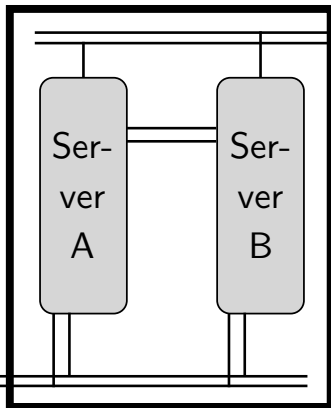
ATC scenario



ATC redundancy

Active server:

- Accepts sensor data
- Provides ASP
- Sends "alive" messages

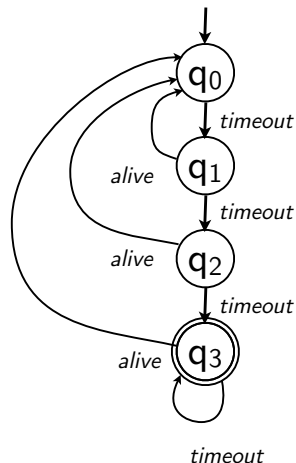


Passive server

- Ignores sensor data
- Monitors "alive" messages
- Takes over in case of failure

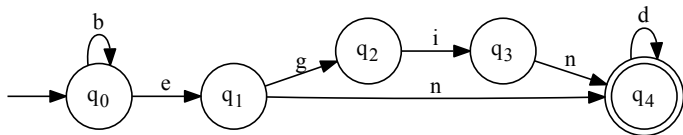
Sensors

Finite automaton to the rescue



- ▶ Two events (“letters”)
 - ▶ **timeout**: 0.1 seconds have passed
 - ▶ **alive**: message from active server
- ▶ States q_0, q_1, q_2 : Server is passive
 - ▶ No processing of input
 - ▶ No sending of alive messages
- ▶ State q_3 : Server becomes active
 - ▶ Process input, provide output to ATC
 - ▶ Send alive messages every 0.1 seconds

Exercise: Finite automaton



Does this automaton accept the words *begin*, *end*, *bind*, *bend*?

Turing Machine

“Universal computer”

- ▶ Very simple model of a computer
 - ▶ Infinite tape, one read/write head
 - ▶ Tape can store letters from a alphabet
 - ▶ Finite automaton controls read/write and movement operations
- ▶ Very powerful model of a computer
 - ▶ Can compute anything any real computer can compute
 - ▶ Can compute anything an “ideal” real computer can compute
 - ▶ Can compute everything a human can compute (?)



Formal grammars

Formalism to **generate** (rather than accept) words over alphabet

terminal symbols: may appear in the produced word (alphabet)

non-terminal symbols: may not appear in the produced word
(temporary symbols)

production rules: $l \rightarrow r$ means: l can be replaced by r anywhere
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Example

Grammar for arithmetic expressions over $\{0, 1\}$

$$\begin{aligned}\Sigma &= \{0, 1, +, \cdot, (,)\} \\ N &= \{E\} \\ P &= \{E \rightarrow 0, E \rightarrow 1, \\ &\quad E \rightarrow (E) \\ &\quad E \rightarrow E + E \\ &\quad E \rightarrow E \cdot E\}\end{aligned}$$

Exercise: Grammars

Using

- ▶ the non-terminal symbols S, B, D, E, G, I, N
- ▶ the terminal symbols b, d, e, g, i, n
- ▶ the production rules $S \rightarrow BEGIN,$
 $BEG \rightarrow E, IN \rightarrow IND, IN \rightarrow N, EG \rightarrow EGG, GGG \rightarrow B,$
 $B \rightarrow b, D \rightarrow d, E \rightarrow e, G \rightarrow g, I \rightarrow i, N \rightarrow n$

can you generate the words *bend* and *end* starting from the symbol S ?

- ▶ If yes, how many steps do you need?
- ▶ If no, why not?

Questions about formal languages

- ▶ For a given language L , how can we find
 - ▶ a corresponding automaton \mathcal{A}_L ?
 - ▶ a corresponding grammar G_L ?
- ▶ What is the simplest automaton for L ?
 - ▶ “simplest” meaning: weakest possible language class
 - ▶ “simplest” meaning: least number of elements
- ▶ How can we use formal descriptions of languages for compilers?
 - ▶ detecting legal words/reserved words
 - ▶ testing if the structure is legal
 - ▶ understanding the meaning by analysing the structure

More questions about formal languages

Closure properties: if L_1 and L_2 are in a class, does this also hold for

- ▶ the **union** of L_1 and L_2 ,
- ▶ the **intersection** of L_1 and L_2 ,
- ▶ the **concatenation** of L_1 and L_2 ,
- ▶ the **complement** of L_1 ?

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Decision problems: for a word w and languages L_1 and L_2 (given by grammars or automata),

- ▶ does $w \in L_1$ hold?
- ▶ is L_1 finite?
- ▶ is L_1 empty?
- ▶ does $L_1 = L_2$ hold?

Abandon all hope. . .



Example applications for formal languages and automata

- ▶ HTML and web browsers
- ▶ Speech recognition and understanding grammars
- ▶ Dialog systems and AI (Siri, Watson)
- ▶ Regular expression matching
- ▶ Compilers and interpreters of programming languages

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End lecture 1

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Selected Solutions

Definition (Alphabet)

An **alphabet** Σ is a finite, non-empty set of characters (symbols, letters).

$$\Sigma = \{c_1, \dots, c_n\}$$

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Example

- 1 $\Sigma_{\text{bin}} = \{0, 1\}$ can express integers in the binary system.
- 2 The English language is based on $\Sigma_{\text{en}} = \{a, \dots, z, A, \dots, Z\}$.
- 3 $\Sigma_{\text{ASCII}} = \{0, \dots, 127\}$ represents the set of ASCII characters [American Standard Code for Information Interchange] coding letters, digits, and special and control characters.

Alphabets: ASCII code chart

ASCII Code Chart

	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	NUL	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	HT	LF	VT	FF	CR	SO	SI
1	DLE	DC1	DC2	DC3	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS	RS	US
2		!	"	#	\$	%	&	'	()	*	+	,	-	.	/
3	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
4	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	P	Q	R	S	T	U	V	W	X	Y	Z	[\]	^	_
6	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
7	p	q	r	s	t	u	v	w	x	y	z	{		}	~	DEL

Definition (Word)

- ▶ A **word** over the alphabet Σ is a finite sequence (list) of characters of Σ :

$$w = c_1 \dots c_n \quad \text{with} \quad c_1, \dots, c_n \in \Sigma.$$

- ▶ The **empty word** with no characters is written as ε .
- ▶ The set of all words over an alphabet Σ is represented by Σ^* .

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In programming languages, words are often referred to as **strings**.

Example

1 Using Σ_{bin} , we can define the words $w_1, w_2 \in \Sigma_{\text{bin}}^*$:

$$w_1 = 01100 \quad \text{and} \quad w_2 = 11001$$

2 Using Σ_{en} , we can define the word $w \in \Sigma_{\text{en}}^*$:

$$w = \text{example}$$

Properties of words

Definition (Length, character access)

- ▶ The **length** $|w|$ of a word w is the number of characters in w .
- ▶ The **number of occurrences** of a character c in w is denoted as $|w|_c$.
- ▶ The **individual characters** within words are accessed using the terminology $w[i]$ with $i \in \{1, 2, \dots, |w|\}$.

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Example

- ▶ $|\text{example}| = 7$ and $|\varepsilon| = 0$
- ▶ $|\text{example}|_e = 2$ and $|\text{example}|_k = 0$
- ▶ $\text{example}[4] = \text{m}$

Appending words

Definition (Concatenation of words)

For words w_1 and w_2 , the concatenation $w_1 \cdot w_2$ is defined as w_1 followed by w_2 .

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$w_1 \cdot w_2$ is often simply written as w_1w_2 .

Example

Let $w_1 = 01$ and $w_2 = 10$.

Then the following holds:

$$w_1w_2 = 0110 \quad \text{and} \quad w_2w_1 = 1001$$

Iterated concatenation

We denote the set of **natural numbers** $\{0, 1, \dots\}$ by \mathbb{N} .

Definition (Power of a word)

For $n \in \mathbb{N}$, the **n -th power** w^n of a word w concatenates the same word n times:

$$\begin{aligned}w^0 &= \varepsilon \\w^n &= w^{n-1} \cdot w \quad \text{if } n > 0\end{aligned}$$

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Example

Let $w = ab$. Then:

$$\begin{aligned}w^0 &= \varepsilon \\w^1 &= ab \\w^3 &= ababab\end{aligned}$$

Exercise: Operations on words

Given the alphabet $\Sigma = \{a, b, c\}$ and the words

▶ $u = abc$

▶ $v = aa$

▶ $w = cb$

what is denoted by the following expressions?

1 $u^2 \cdot w$

2 $v \cdot \varepsilon \cdot w \cdot u^0$

3 $|u^3|_a$

4 $v \cdot a^2 \cdot (v[4])$

5 $(v \cdot a^2 \cdot v)[4]$

6 $|w^0|$

7 $|w^0 \cdot w|$

Definition (Formal language)

For an alphabet Σ , a **formal language over Σ** is a subset $L \subseteq \Sigma^*$.

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Example

Let $L_{\mathbb{N}} = \{1w \mid w \in \Sigma_{\text{bin}}^*\} \cup \{0\}$.

Then $L_{\mathbb{N}}$ is the set of all words that represent integers using the binary system (all words starting with 1 and the word 0):

$$100 \in L_{\mathbb{N}} \quad \text{but} \quad 010 \notin L_{\mathbb{N}}.$$

Definition (Numeric value)

We define the function

$$n : L_{\mathbb{N}} \rightarrow \mathbb{N}$$

as the function returning the numeric value of a word in the language $L_{\mathbb{N}}$. This means

- (a) $n(0) = 0$,
- (b) $n(1) = 1$,
- (c) $n(w0) = 2 \cdot n(w)$ for $|w| > 0$,
- (d) $n(w1) = 2 \cdot n(w) + 1$ for $|w| > 0$.

Definition (Prime numbers)

We define the language $L_{\mathbb{P}}$ as the language representing prime numbers in the binary system:

$$L_{\mathbb{P}} = \{w \in L_{\mathbb{N}} \mid n(w) \in \mathbb{P}\}.$$

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One way to formally express the set of all prime numbers is

$$\mathbb{P} = \{p \in \mathbb{N}^{\geq 2} \mid \{t \in \mathbb{N} \mid \exists k \in \mathbb{N} : k \cdot t = p\} = \{1, p\}\}.$$

C functions as a language

Definition

We define the language $L_C \subset \Sigma_{\text{ASCII}}^*$ as the set of all C function definitions with a declaration of the form:

$$\text{char* } f(\text{char* } x);$$

(where f and x are legal C identifiers).

Then L_C contains the ASCII code of all those definitions of C functions processing and returning a string.

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(where f and x are legal C identifiers).

Then L_C contains the ASCII code of all those definitions of C functions processing and returning a string.

Examples

- ▶ $\text{char* } f(\text{char* } x) \{ \text{return } x; \} \in L_C$
- ▶ $\text{char* } f(\text{char* } x) \{ \text{return } ""; \} \in L_C$
- ▶ $\text{char* } f(\text{char* } x, \text{int } y) \{ \text{return } ""; \} \notin L_C$
- ▶ $\text{Harakiri} \notin L_C$

C function evaluations as a language

Definition

Using the alphabet $\Sigma_{\text{ASCII}+} = \Sigma_{\text{ASCII}} \cup \{\dagger\}$, we define the **universal language**

$$L_u = \{f\dagger x\dagger y \mid \text{(a) and (b) and (c)}\} \quad \text{with}$$

- (a) $f \in L_C$ (i.e. f is a C function mapping strings to strings),
- (b) $x, y \in \Sigma_{\text{ASCII}}^*$,
- (c) applying f to x terminates and returns y .

C function evaluations as a language

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- (c) applying f to x terminates and returns y .

Examples

- ▶ `char* f(char* x) {return x;}` $\dagger \text{aaa} \dagger \text{aaa} \in L_u$
- ▶ `char* f(char* x) {return x;}` $\dagger \text{aaa} \dagger \text{bbb} \notin L_u$
- ▶ `char* f(char* x) {return "";}` $\dagger \text{aaa} \dagger \in L_u$

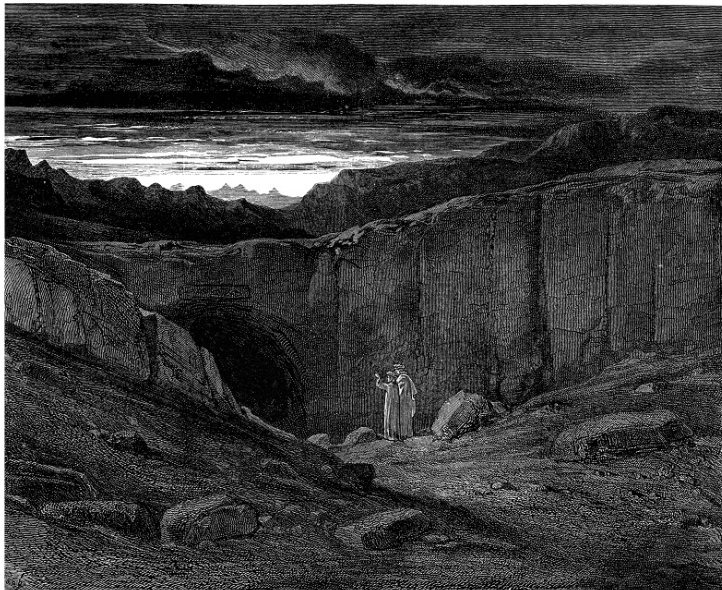
Formal languages can be highly complex

Formal languages have a wide scope:

- ▶ Testing whether a word belongs to $L_{\mathbb{N}}$ is straightforward
- ▶ The same test for $L_{\mathbb{P}}$ is more complex
- ▶ The test for L_C requires a proper C parser
- ▶ Later we will see that there is no algorithm that will correctly perform the membership test for L_u

```
def inN(x):  
    if x=="0":  
        return True  
    if x=="":  
        return False  
    if x[0]!="1":  
        return False  
    for c in x[1:]:  
        if c!="0" and c!="1":  
            return False  
    return True
```

Abandon all hope. . .



Definition (Product of formal languages)

Given an alphabet Σ and the formal languages $L_1, L_2 \subseteq \Sigma^*$, we define the **product**

$$L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}.$$

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Example

Using the alphabet Σ_{en} , we define the languages

$$L_1 = \{ab, bc\} \quad \text{and} \quad L_2 = \{ac, cb\}.$$

The product is

$$L_1 \cdot L_2 = \{abac, abcb, bcac, bccb\}.$$

Definition (Power of a language)

Given an alphabet Σ , a formal language $L \subseteq \Sigma^*$, and an integer $n \in \mathbb{N}$, we define the n -th power of L (recursively) as follows:

$$L^0 = \{\varepsilon\}$$

$$L^n = L^{n-1} \cdot L$$

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$$\begin{aligned}L^0 &= \{\varepsilon\} \\L^n &= L^{n-1} \cdot L\end{aligned}$$

Example

Using the alphabet Σ_{en} , we define the language $L = \{ab, ba\}$. Thus:

$$\begin{aligned}L^0 &= \{\varepsilon\} \\L^1 &= \{\varepsilon\} \cdot \{ab, ba\} = \{ab, ba\} \\L^2 &= \{ab, ba\} \cdot \{ab, ba\} = \{abab, abba, baab, baba\}\end{aligned}$$

The Kleene Star operator

Definition (Kleene Star)

Given an alphabet Σ and a formal language $L \subseteq \Sigma^*$, we define the **Kleene star** operator as

$$L^* = \bigcup_{n \in \mathbb{N}} L^n.$$

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Given an alphabet Σ and a formal language $L \subseteq \Sigma^*$, we define the **Kleene star** operator as

$$L^* = \bigcup_{n \in \mathbb{N}} L^n.$$

Example

Using the alphabet Σ_{en} , we define the language $L = \{a\}$. Thus:

$$L^* = \{a^n \mid n \in \mathbb{N}\}.$$

Exercise: formal languages

Given

- ▶ the alphabet Σ_{bin} ,
- ▶ the function $n(w)$ mapping binary words to numbers as defined on page 39, and
- ▶ the language $L = \{1\}$,

formally describe the following:

- a) the language $M = L^* \setminus \{\varepsilon\}$
- b) the set $N = \{n(w) \mid w \in M\}$
- c) the language $M^- = \{w \mid n(w) - 1 \in N\}$
- d) the language $M^+ = \{w \mid n(w) + 1 \in N\}$

Introduction

Regular Languages and Finite Automata

Regular Expressions

Finite Automata

The Pumping Lemma

Properties of Regular Languages

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

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- ▶ Characterize tokens for compilers
- ▶ Describe search terms for a database
- ▶ Filter through genomic data
- ▶ Extract URLs from web pages
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Regular expressions

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- ▶ Characterize tokens for compilers
- ▶ Describe search terms for a database
- ▶ Filter through genomic data
- ▶ Extract URLs from web pages
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**The set of all regular expressions (over an alphabet)
is a formal language**

Each single regular expression describes a formal language

Introductory Examples

Consider $\Sigma_{\text{bin}} = \{0, 1\}$. With regular expressions we can conveniently and rigorously describe many languages:

▶ $1(0 + 1)^*$

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- ▶ 11^*

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- ▶ 11^* – all words consisting of one or more letters 1 (M from the last exercise)
- ▶ $0(10)^* + (10)^* + 1(01)^* + (01)^*$

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- ▶ 11^* – all words consisting of one or more letters 1 (M from the last exercise)
- ▶ $0(10)^* + (10)^* + 1(01)^* + (01)^*$ – the language of all words where no two subsequent characters are the same

Reminder: Power sets

Definition (Power set of a set)

- ▶ Assume a set S . Then the power set of S , written as 2^S , is the set of all subsets of S .
- ▶ In particular, if Σ is an alphabet, 2^{Σ^*} is the set of all subsets of Σ^* and hence the set of all possible formal languages over Σ .

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Example

$$2^{\Sigma_{\text{bin}}} = 2^{\{0,1\}} = \{\{\}, \{0\}, \{1\}, \{0, 1\}\},$$

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Example

$$\begin{aligned}2^{\Sigma_{\text{bin}}} &= 2^{\{0,1\}} = \{\{\}, \{0\}, \{1\}, \{0, 1\}\}, \\2^{\Sigma_{\text{bin}}^*} &= 2^{\{\epsilon, 0, 1, 00, 01, \dots\}}\end{aligned}$$

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Example

$$\begin{aligned}2^{\Sigma_{\text{bin}}} &= 2^{\{0,1\}} = \{\{\}, \{0\}, \{1\}, \{0, 1\}\}, \\2^{\Sigma_{\text{bin}}^*} &= 2^{\{\epsilon, 0, 1, 00, 01, \dots\}} \\&= \{\{\}, \{\epsilon\}, \{0\}, \{1\}, \{00\}, \{01\}, \dots \\&\quad \dots \{\epsilon, 0\}, \{\epsilon, 1\}, \{\epsilon, 00\}, \{\epsilon, 01\}, \dots \\&\quad \dots \{010, 1110, 10101\}, \dots\}.\end{aligned}$$

Regular expressions and formal languages

A regular expression over Σ ...

- ▶ ... is a word over the extended alphabet $\Sigma \cup \{\emptyset, \varepsilon, +, \cdot, *, (,)\}$
 - ▶ Note that we implicitly assume that $\{\emptyset, \varepsilon, +, \cdot, *, (,)\} \cap \Sigma = \{\}$
 - ▶ \emptyset denotes a regular expression (syntax)
 - ▶ $\{\}$ denotes the empty set (semantics)
- ▶ ... describes a formal language over Σ

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Terminology

The following terms are defined on the next slides:

- ▶ R_Σ is the set of all regular expressions over the alphabet Σ .
- ▶ The function $L : R_\Sigma \rightarrow 2^{\Sigma^*}$ assigns a formal language $L(r) \subseteq \Sigma^*$ to each regular expression r .

Definition (Regular expressions)

The set of regular expressions R_Σ over the alphabet Σ is defined as follows:

- 1 The regular expression \emptyset denotes the **empty language**.
 $\emptyset \in R_\Sigma$ and $L(\emptyset) = \{\}$
- 2 The regular expression ε denotes the language containing only the empty word.
 $\varepsilon \in R_\Sigma$ and $L(\varepsilon) = \{\varepsilon\}$
- 3 Each symbol in the alphabet Σ is a regular expression.
 $c \in \Sigma \Rightarrow c \in R_\Sigma$ and $L(c) = \{c\}$

Definition (Regular expressions (cont'))

- 4 The operator $+$ denotes the **union** of the languages of r_1 and r_2 .
 $r_1 \in R_\Sigma, r_2 \in R_\Sigma \Rightarrow r_1 + r_2 \in R_\Sigma$ and $L(r_1 + r_2) = L(r_1) \cup L(r_2)$
- 5 The operator \cdot denotes the **product** of the languages of r_1 and r_2 .
 $r_1 \in R_\Sigma, r_2 \in R_\Sigma \Rightarrow r_1 \cdot r_2 \in R_\Sigma$ and $L(r_1 \cdot r_2) = L(r_1) \cdot L(r_2)$
- 6 The **Kleene star** of a regular expression r denotes the Kleene star of the language of r .
 $r \in R_\Sigma \Rightarrow r^* \in R_\Sigma$ and $L(r^*) = (L(r))^*$
- 7 **Brackets** can be used to group regular expressions without changing their language.
 $r \in R_\Sigma \Rightarrow (r) \in R_\Sigma$ and $L((r)) = L(r)$

Equivalence of regular expressions

Definition (Equivalence and precedence)

- ▶ Two regular expressions r_1 and r_2 are **equivalent** if they denote the same language: $r_1 \doteq r_2$ if and only if $L(r_1) = L(r_2)$
- ▶ The operators have the following **precedence**:
 $(\dots) > * > \cdot > +$
- ▶ The product operator \cdot can be omitted.

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Example

$$\begin{aligned} a + b \cdot c^* &\doteq a + (b \cdot (c^*)) \\ ac + bc^* &\doteq a \cdot c + b \cdot c^* \end{aligned}$$

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$$\begin{aligned}a + b \cdot c^* &\doteq a + (b \cdot (c^*)) \\ac + bc^* &\doteq a \cdot c + b \cdot c^*\end{aligned}$$

Note: Some authors (and tools) use $|$ as the union operator.

Examples for regular expressions

Example

Let $\Sigma_{abc} = \{a, b, c\}$.

- ▶ The regular expression $r_1 = (a + b + c)(a + b + c)$ describes all the words of exactly two symbols:

$$L(r_1) = \{w \in \Sigma_{abc}^* \mid |w| = 2\}$$

- ▶ The regular expression $r_2 = (a + b + c)(a + b + c)^*$ describes all the words of one or more symbols:

$$L(r_2) = \{w \in \Sigma_{abc}^* \mid |w| \geq 1\}$$

Exercise: regular expressions

- 1 Using the alphabet $\Sigma_{abc} = \{a, b, c\}$, give a regular expression r_1 for all the words $w \in \Sigma_{abc}^*$ containing exactly one a or exactly one b .
- 2 Formally describe $L(r_1)$ as a set.
- 3 Using the alphabet $\Sigma_{abc} = \{a, b, c\}$, give a regular expression r_2 for all the words containing at least one a and at least one b .
- 4 Using the alphabet $\Sigma_{bin} = \{0, 1\}$, give a regular expression for all the words whose third last symbol is 1 .
- 5 Using the alphabet Σ_{bin} , give a regular expression for all the words not containing the string 110 .
- 6 Which language is described by the regular expression

$$r_6 = (1 + \varepsilon)(00^*1)^*0^*?$$

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- 6 Which language is described by the regular expression

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Theorem

- 1 $r_1 + r_2 \doteq r_2 + r_1$ (*commutative law*)
- 2 $(r_1 + r_2) + r_3 \doteq r_1 + (r_2 + r_3)$ (*associative law*)
- 3 $(r_1 r_2) r_3 \doteq r_1 (r_2 r_3)$ (*associative law*)
- 4 $\emptyset r \doteq \emptyset$
- 5 $\varepsilon r \doteq r$
- 6 $\emptyset + r \doteq r$
- 7 $(r_1 + r_2) r_3 \doteq r_1 r_3 + r_2 r_3$ (*distributive law*)
- 8 $r_1 (r_2 + r_3) \doteq r_1 r_2 + r_1 r_3$ (*distributive law*)

Proof of some rules

Proof of Rule 1 ($r_1 + r_2 \doteq r_2 + r_1$).

$$L(r_1 + r_2) = L(r_1) \cup L(r_2) = L(r_2) \cup L(r_1) = L(r_2 + r_1)$$



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Proof of Rule 4 ($\emptyset r \doteq \emptyset$).

$$L(\emptyset r) \quad \text{Def. } \underline{\text{concat}} \quad L(\emptyset) \cdot L(r)$$

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Proof of Rule 4 ($\emptyset r \doteq \emptyset$).

$$\begin{array}{lcl} L(\emptyset r) & \text{Def. } \underline{\text{concat}} & L(\emptyset) \cdot L(r) \\ & \text{Def. } \underline{\text{empty regex}} & \{\} \cdot L(r) \end{array}$$

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Proof of some rules

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Theorem

9 $r + r \doteq r$

10 $(r^*)^* \doteq r^*$

11 $\emptyset^* \doteq \varepsilon$

12 $\varepsilon^* \doteq \varepsilon$

13 $r^* \doteq \varepsilon + r^*r$

14 $r^* \doteq (\varepsilon + r)^*$

15 $\varepsilon \notin L(s)$ and $r \doteq rs + t \longrightarrow r \doteq ts^*$ (proof by Arto Salomaa)

16 $r^*r \doteq rr^*$ (see Lemma: Kleene Star below)

17 $\varepsilon \notin L(s)$ and $r \doteq sr + t \longrightarrow r \doteq s^*t$ (Arden's Lemma)

Lemma: Kleene Star (1)

Lemma (Kleene Star)

$$u^* u \doteq uu^*$$

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Proof of Case 1: $\varepsilon \notin L(u)$.

$$u^*u \doteq (\varepsilon + u^*u)u \quad (\text{by 13. } (r)^* \doteq \varepsilon + (r)^*r)$$

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$$\begin{aligned} u^*u &\doteq (\varepsilon + u^*u)u && \text{(by 13. } (r)^* \doteq \varepsilon + (r)^*r) \\ &\doteq (u^*u + \varepsilon)u && \text{(by 1. } r_1 + r_2 \doteq r_2 + r_1) \end{aligned}$$

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□

Lemma: Kleene Star (2)

Proof of Case 2: $\varepsilon \in L(u)$.

We show $L(u^*u) = L(u^*) = L(uu^*)$

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Proof of Case 2: $\varepsilon \in L(u)$.

We show $L(u^*u) = L(u^*) = L(uu^*)$

a) Proof of $L(u^*u) \subseteq L(u^*)$

$$L(u^*u) = L(u^*) \cdot L(u)$$

$$= (L(u))^* \cdot L(u)$$

$$= \left(\bigcup_{i \geq 0} L(u)^i\right) \cdot L(u)$$

$$= \bigcup_{i \geq 0} (L(u)^i \cdot L(u))$$

$$= \bigcup_{i \geq 1} L(u)^i$$

$$\subseteq L(u^*)$$

Lemma: Kleene Star (2)

Proof of Case 2: $\varepsilon \in L(u)$.

We show $L(u^*u) = L(u^*) = L(uu^*)$

a) Proof of $L(u^*u) \subseteq L(u^*)$

$$\begin{aligned}L(u^*u) &= L(u^*) \cdot L(u) \\&= (L(u))^* \cdot L(u) \\&= \left(\bigcup_{i \geq 0} L(u)^i\right) \cdot L(u) \\&= \bigcup_{i \geq 0} (L(u)^i \cdot L(u)) \\&= \bigcup_{i \geq 1} L(u)^i \\&\subseteq L(u^*)\end{aligned}$$

b) Proof of $L(u^*u) \supseteq L(u^*)$

$$\begin{aligned}L(u^*u) &= \{tv \mid t \in L(u^*), v \in L(u)\} \\&\supseteq \{tv \mid t \in L(u^*), v = \varepsilon\} \\&= \{t \mid t \in L(u^*)\} \\&= L(u^*)\end{aligned}$$

Lemma: Kleene Star (2)

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We show $L(u^*u) = L(u^*) = L(uu^*)$

a) Proof of $L(u^*u) \subseteq L(u^*)$

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b) Proof of $L(u^*u) \supseteq L(u^*)$

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- ▶ a) and b) imply $L(u^*u) = L(u^*)$
- ▶ $L(uu^*) = L(u^*)$: strictly analogous



A note on Arto/Arden

- ▶ Arto: $\varepsilon \notin L(s)$ and $r \doteq rs + t \longrightarrow r \doteq ts^*$
- ▶ Why do we need $\varepsilon \notin L(s)$?
 - ▶ This guarantees that **only** words of the form ts^* are in $L(r)$
 - ▶ Example: $r \doteq rs + t$ mit $s = b^*$, $t = a$.
 - ▶ If we could apply Arto, the result would be $r \doteq a(b^*)^* \doteq ab^*$
 - ▶ But $L = \{ab^*\} \cup \{b^*\}$ also fulfills the equation, i.e. there is no single unique solution in this case
 - ▶ Intuitively: $\varepsilon \in L(s)$ is a second escape from the recursion that bypasses t
- ▶ The case for Arden's lemma ($\varepsilon \notin L(s)$ and $r \doteq sr + t \longrightarrow r \doteq s^*t$) is analogous

Exercise: Algebra on regular expressions

- 1 Prove the equivalence using only algebraic operations:

$$r^* \doteq \varepsilon + r^*.$$

- 2 Simplify the regular expression s using only algebraic operations:

$$s = 0(\varepsilon + 0 + 1)^* + (\varepsilon + 1)(1 + 0)^* + \varepsilon.$$

- 3 Prove the equivalence using only algebraic equivalences:

$$10(10)^* \doteq 1(01)^*0.$$

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Solutions

Exercise: Algebra on regular expressions

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Solutions

End lecture 3

Outline

Introduction

Regular Languages and Finite Automata

Regular Expressions

Finite Automata

The Pumping Lemma

Properties of Regular Languages

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

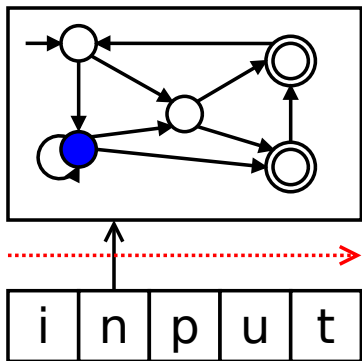
Bonus Exercises

Selected Solutions

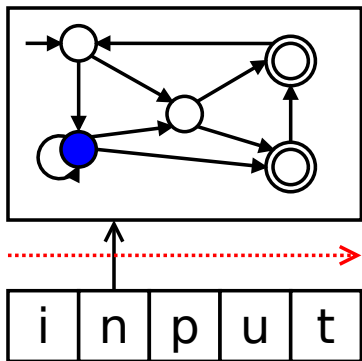
Finite Automata: Motivation

- ▶ Simple model of computation
- ▶ Can recognize **regular languages**
- ▶ Equivalent to regular expressions
 - ▶ We can automatically generate a FA from a RE
 - ▶ We can automatically generate an RE from an FA
- ▶ Two variants:
 - ▶ Deterministic (DFA, now)
 - ▶ Non-deterministic (NFA, later)
- ▶ Easy to implement in actual programs

Deterministic Finite Automata: Idea



Deterministic Finite Automata: Idea



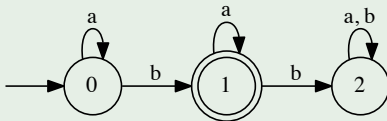
Deterministic finite automaton (DFA)

- ▶ is in one of finitely many **states**
- ▶ starts in the **initial state**
- ▶ processes **input** from left to right
 - ▶ changes state depending on character read
 - ▶ determined by **transition function**
 - ▶ no rewinding!
 - ▶ no writing!
- ▶ accepts input if
 - ▶ after reading the entire input
 - ▶ a **final state** is reached

DFA \mathcal{A} for a^*ba^*

Example (Automaton \mathcal{A})

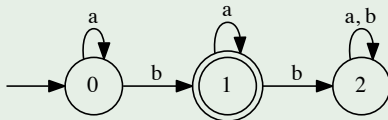
\mathcal{A} is a simple DFA recognizing the regular language a^*ba^* .



DFA \mathcal{A} for a^*ba^*

Example (Automaton \mathcal{A})

\mathcal{A} is a simple DFA recognizing the regular language a^*ba^* .



- ▶ \mathcal{A} has three **states**, 0, 1 and 2.
- ▶ It operates on the **alphabet** $\{a, b\}$.
- ▶ The **transition function** is indicated by the arrows.
- ▶ 0 is the **initial** state (with an arrow “pointing at it from anywhere”).
- ▶ 1 is an **accepting** state (represented as a double circle).
- ▶ 2 is also called a **junk** state (once it is reached, the word can never be accepted).

DFA: formal definition

Definition (Deterministic Finite Automaton)

A **deterministic finite automaton** (DFA) is a quintuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ with the following components

- ▶ Q is a finite set of **states**.
- ▶ Σ is the (finite) input **alphabet**.
- ▶ $\delta : Q \times \Sigma \rightarrow Q$ is the **transition function**.
- ▶ $q_0 \in Q$ is the **initial** state.
- ▶ $F \subseteq Q$ is the set of final (or **accepting**) states.

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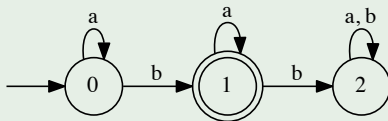
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Notes:

- ▶ δ is a **total** function – there has to be a transition from every state for every letter
- ▶ ... but automata with **partial** transition functions can be “repaired” into a proper DFA by adding a junk state

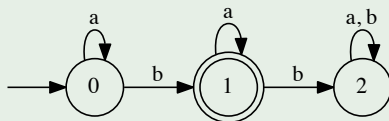
Formal definition of \mathcal{A}

Example



Formal definition of \mathcal{A}

Example



\mathcal{A} is expressed as $(Q, \Sigma, \delta, q_0, F)$ with

- ▶ $Q = \{0, 1, 2\}$
- ▶ $\Sigma = \{a, b\}$
- ▶ $\delta(0, a) = 0; \delta(0, b) = 1, \delta(1, a) = 1; \delta(1, b) = \delta(2, a) = \delta(2, b) = 2$
- ▶ $q_0 = 0$
- ▶ $F = \{1\}$

Definition (Language accepted by an automaton)

The state transition function δ is generalised to a function δ' whose second argument is a word as follows:

- ▶ $\delta' : Q \times \Sigma^* \rightarrow Q$
- ▶ $\delta'(q, \varepsilon) = q$ for every $q \in Q$
- ▶ $\delta'(q, wc) = \delta(\delta'(q, w), c)$ with $c \in \Sigma; w \in \Sigma^*$

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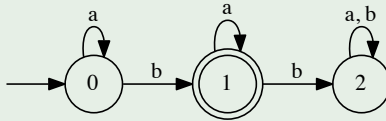
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The **language accepted by a DFA** $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ is defined as

$$L(\mathcal{A}) = \{w \in \Sigma^* \mid \delta'(q_0, w) \in F\}.$$

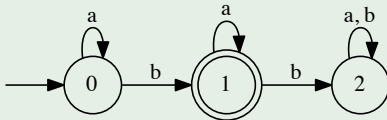
Language accepted by \mathcal{A}

Example



Language accepted by \mathcal{A}

Example



- ▶ $\delta'(0, aa) = \delta(\delta'(0, a), a) = \delta(\delta(\delta'(0, \varepsilon), a), a) = 0$
- ▶ $\delta'(1, aaa) = 1$
- ▶ $\delta'(0, bb) = \delta'(1, b) = 2$
- ▶ $L(\mathcal{A}) = \{w \in \{a, b\}^* \mid w = a^n b a^m \text{ and } n, m \in \mathbb{N}\}$

Run of a DFA

Definition (Configuration, Run)

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

A **configuration** of \mathcal{A} is a pair (q, w) with $q \in Q$ and $w \in \Sigma^*$.

Run of a DFA

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A **configuration** of \mathcal{A} is a pair (q, w) with $q \in Q$ and $w \in \Sigma^*$.

A **run** of \mathcal{A} on a word $w = c_1 \cdot c_2 \cdots c_n$ is a sequence of configurations:

$$((q_0, c_1 \cdot c_2 \cdots c_n), (q_1, c_2 \cdots c_n), \dots, (q_n, \varepsilon))$$

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where

- ▶ $q_i \in Q$ holds for $1 \leq i \leq n$ and
- ▶ $\delta(q_{i-1}, c_i) = q_i$ holds for $1 \leq i \leq n$.

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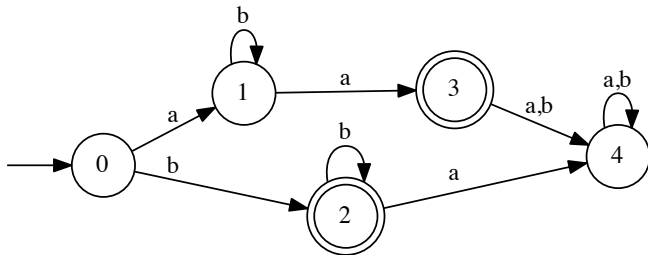
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- ▶ $\delta(q_{i-1}, c_i) = q_i$ holds for $1 \leq i \leq n$.

A run is **accepting** if $q_n \in F$ holds.

The language accepted by \mathcal{A} can alternatively be defined as the set of all words for which there exists an accepting run of \mathcal{A} .

Exercise: DFA

1 Given this graphical representation of a DFA \mathcal{A} :



- Give a regular expression describing $L(\mathcal{A})$.
- Give a formal definition of \mathcal{A} .

Exercise: DFA

2 Give

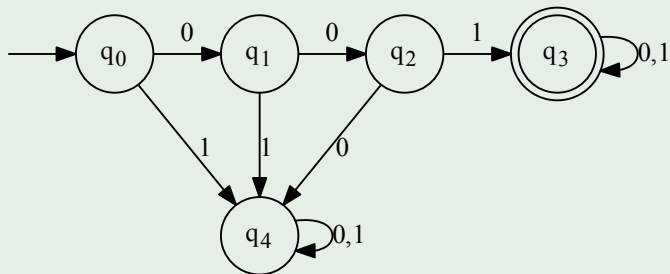
- ▶ a regular expression,
- ▶ a graphical representation of a DFA, and
- ▶ a formal definition

for the language $L \subset \{a, b\}^*$ containing all those words featuring the substring ab

- a) at the beginning,
- b) at an arbitrary position,
- c) at the end.

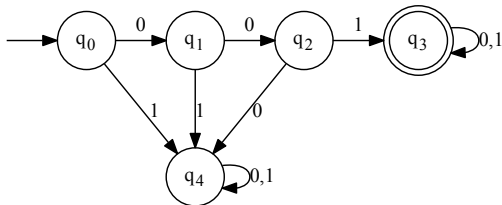
Another example

Example



Which language is recognized by the DFA?

Tabular representation of a DFA

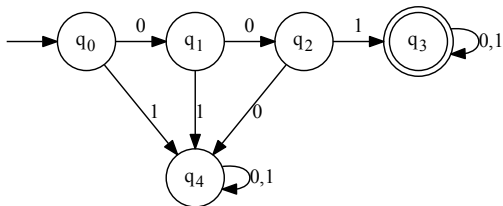


$$\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$$

- ▶ $Q = \{q_0, q_1, q_2, q_3, q_4\}$
- ▶ $\Sigma = \{0, 1\}$
- ▶ Initial state: q_0
- ▶ $F = \{q_3\}$

δ	0	1
q_0	q_1	q_4
q_1	q_2	q_4
q_2	q_4	q_3
q_3	q_3	q_3
q_4	q_4	q_4

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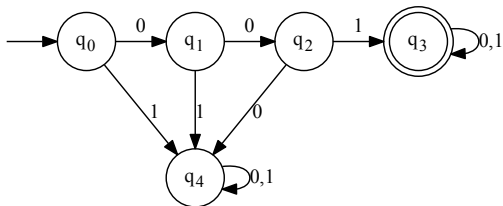


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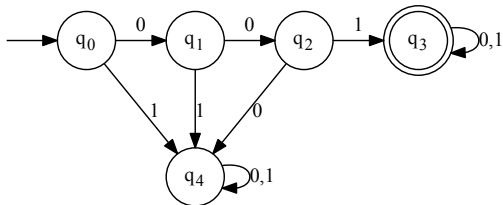
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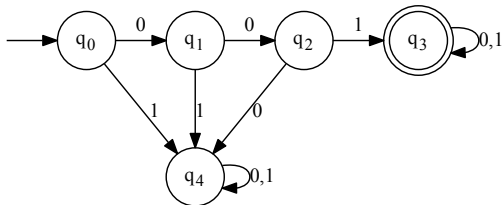
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	δ	0	1
\rightarrow	q_0	q_1	q_4
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	q_2	q_4	q_3
*	q_3	q_3	q_3
	q_4	q_4	q_4

Tabular representation of a DFA



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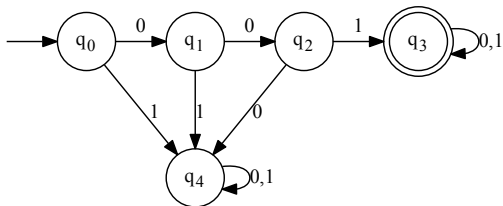
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		δ		
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→	q_0	q_1	q_4	
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Tabular representation of a DFA



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	δ	0	1
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	q_1	q_2	q_4
	q_2	q_4	q_3
*	q_3	q_3	q_3
	q_4	q_4	q_4

DFA: Tabular representation in practice

Delta		0	1
-> q0		q1	q4
q1		q2	q4
q2		q4	q3
* q3		q3	q3
q4		q4	q4

DFA: Tabular representation in practice

Delta		0	1

-> q0		q1	q4
q1		q2	q4
q2		q4	q3
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```
> easim.py fsa001.txt 10101
Processing: 10101
q0 :: 1 -> q4
q4 :: 0 -> q4
q4 :: 1 -> q4
q4 :: 0 -> q4
q4 :: 1 -> q4
Rejected
```

DFA: Tabular representation in practice

Delta		0	1

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q1		q2	q4
q2		q4	q3
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q4		q4	q4

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q4 :: 0 -> q4
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Rejected
```

```
> easim.py fsa001.txt 101
Processing: 101
q0 :: 1 -> q4
q4 :: 0 -> q4
q4 :: 1 -> q4
Rejected
```

DFAs in tabular form: exercise

- ▶ Give the following DFA ...
 - ▶ as a formal 5-tuple
 - ▶ as a diagram

parity		0	1

-> even		even	odd
* odd		odd	even

- ▶ Characterize the language accepted by the DFA

- ▶ Assume
 - ▶ $\Sigma = \{a, b, c\}$
 - ▶ $L_1 = \{ubw \mid u \in \Sigma^*, w \in \Sigma\}$
 - ▶ $L_2 = \{ubw \mid u \in \Sigma, w \in \Sigma^*\}$
- ▶ Group 1 (your family name starts with A-M):
Find a DFA \mathcal{A} with $L(\mathcal{A}) = L_1$
- ▶ Group 2 (your family name does not start with A-M):
Find a DFA \mathcal{A} with $L(\mathcal{A}) = L_2$

Introduction

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Non-Determinism

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Drawbacks of deterministic automata

Deterministic automata:

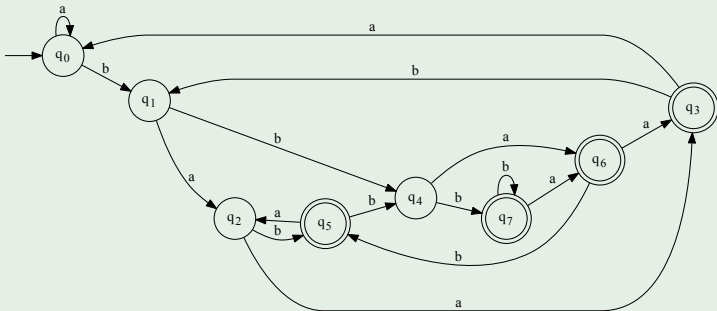
- ▶ Transition function δ
 - ▶ exactly one transition from every configuration
- ▶ can be complex even for simple languages

Drawbacks of deterministic automata

Deterministic automata:

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 - ▶ exactly one transition from every configuration
- ▶ can be complex even for simple languages

Example (DFA \mathcal{A} for $(a + b)^*b(a + b)(a + b)$)



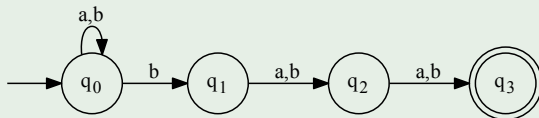
Non-Determinism

- ▶ FA can be simplified if one input can lead to
 - ▶ one transition,
 - ▶ multiple transitions, or
 - ▶ no transition.
- ▶ Intuitively, such an FA selects its next state from a set of states depending on the current state and the input
 - ▶ and always chooses the “right” one
- ▶ This is called a **non-deterministic finite automaton** (NFA)

Non-Determinism

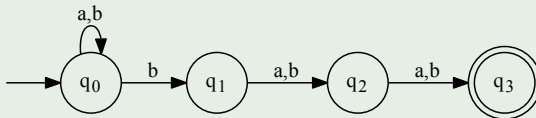
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Example (NFA \mathcal{B} for $(a + b)^*b(a + b)(a + b)$)



Non-Deterministic automata

Example (Transitions in \mathcal{B})



- ▶ In state q_0 with input b , the FA has to “guess” the next state.
- ▶ The string $abab$ can be read in three ways:
 - 1 $q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_0$ (failure)
 - 2 $q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_1$ (failure)
 - 3 $q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_3$ (success)
- ▶ An NFA accepts an input w if there **exists** an accepting run on w !

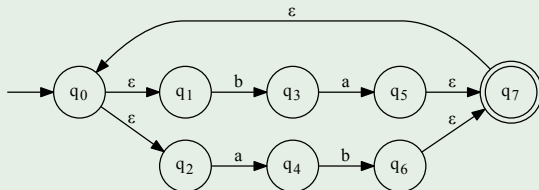
NFA: non-deterministic transitions and ε -transitions

- ▶ Non-deterministic transitions allow an NFA to go to more than one successor state
 - ▶ Instead of a **function** δ , an NFA has a transition **relation** Δ
- ▶ An NFA can additionally change its current state without reading an input symbol: $q_1 \xrightarrow{\varepsilon} q_2$.
 - ▶ This is called a **spontaneous transition** or **ε -transition**
 - ▶ Thus, Δ is a relation on $Q \times (\Sigma \cup \{\varepsilon\}) \times Q$

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Example (NFA with ε -transitions)



Definition (NFA)

An **NFA** is a quintuple $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$ with the following components:

- 1 Q is the finite set of states.
- 2 Σ is the input alphabet.
- 3 Δ is a **relation** over $Q \times (\Sigma \cup \{\varepsilon\}) \times Q$.
- 4 $q_0 \in Q$ is the initial state.
- 5 $F \subseteq Q$ is the set of final states.

Run of a nondeterministic automaton

Definition (Configuration, Run of an NFA)

Let $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$ be an NFA.

A **configuration** of \mathcal{A} is a pair (q, w) as for a DFA.

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A **run** of \mathcal{A} on a word $w_0 = c_1 \cdot c_2 \cdots c_n$ is a sequence of transitions

$$r = ((q_0, w_0), (q_1, w_1), \dots, (q_m, \varepsilon))$$

such that the following conditions are satisfied:

- ▶ $q_i \in Q$ for all $1 \leq i \leq m$,
- ▶ if r contains the configurations $(q_i, w_i), (q_{i+1}, w_{i+1})$, then
 - ▶ there is a transition $(q_i, c, q_{i+1}) \in \Delta$ with $c \in \Sigma \cup \{\varepsilon\}$
 - ▶ $w_i = c \cdot w_{i+1}$.

Run of a nondeterministic automaton

Definition (Configuration, Run of an NFA)

Let $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$ be an NFA.

A **configuration** of \mathcal{A} is a pair (q, w) as for a DFA.

A **run** of \mathcal{A} on a word $w_0 = c_1 \cdot c_2 \cdots c_n$ is a sequence of transitions

$$r = ((q_0, w_0), (q_1, w_1), \dots, (q_m, \varepsilon))$$

such that the following conditions are satisfied:

- ▶ $q_i \in Q$ for all $1 \leq i \leq m$,
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The slightly more complex definition is necessary to handle ε -transitions.

Language recognized by an NFA

Definition (Language recognized by an NFA)

Let $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$ be an NFA. The language accepted by \mathcal{A} is

$$L(\mathcal{A}) = \{w \mid \text{there is an accepting run of } \mathcal{A} \text{ on } w\}$$

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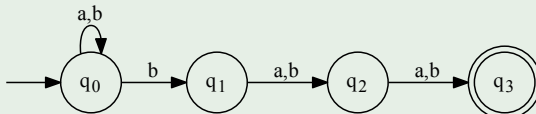
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Note:

- ▶ Only existence of one accepting run is required
- ▶ It does not matter if there are also non-accepting runs on w

Example: NFA definition

Example (Formal definition of \mathcal{B})



$\mathcal{B} = (Q, \Sigma, \Delta, q_0, F)$ with

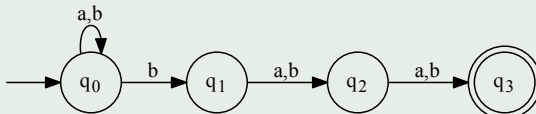
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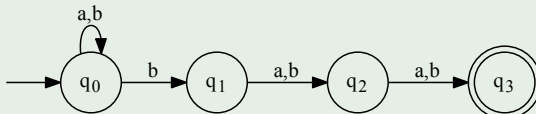
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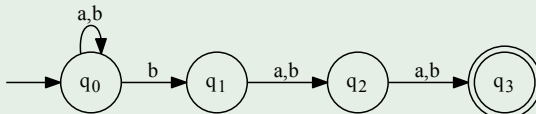
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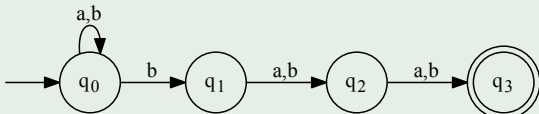
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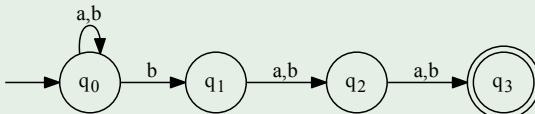
$\Sigma = \{a, b\}$

$F = \{q_3\}$

$\Delta = \{(q_0, a, q_0), (q_0, b, q_0), (q_0, b, q_1),$
 $(q_1, a, q_2), (q_1, b, q_2),$
 $(q_2, a, q_3), (q_2, b, q_3)\}$

Example: NFA definition

Example (Formal definition of \mathcal{B})



$\mathcal{B} = (Q, \Sigma, \Delta, q_0, F)$ with

$Q = \{q_0, q_1, q_2, q_3\}$

$\Sigma = \{a, b\}$

$F = \{q_3\}$

Δ	a	b	ϵ
q_0	$\{q_0\}$	$\{q_0, q_1\}$	$\{\}$
q_1	$\{q_2\}$	$\{q_2\}$	$\{\}$
q_2	$\{q_3\}$	$\{q_3\}$	$\{\}$
q_3	$\{\}$	$\{\}$	$\{\}$

Exercise: NFA

Develop an NFA \mathcal{A} whose language $L(\mathcal{A}) \subset \{a, b\}^*$ contains all those words featuring the substring aba . Give:

- ▶ a regular expression representing $L(\mathcal{A})$,
- ▶ a graphical representation of \mathcal{A} ,
- ▶ a formal definition of \mathcal{A} .

Equivalence of DFA and NFA

Theorem (Equivalence of DFA and NFA)

NFAs and DFAs recognize the same class of languages.

- ▶ *For every DFA \mathcal{A} there is an NFA \mathcal{B} with $L(\mathcal{A}) = L(\mathcal{B})$.*
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- ▶ The direction DFA to NFA is trivial:
 - ▶ Every DFA is (essentially) an NFA
 - ▶ ... since every function is a relation
- ▶ What about the other direction?

Equivalence of DFA and NFA

Equivalence of DFAs and NFAs can be shown by transforming

- ▶ an NFA \mathcal{A}
- ▶ into a DFA $\text{det}(\mathcal{A})$ accepting the same language.

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Method:

- ▶ states of $\det(\mathcal{A})$ represent **sets of states** of \mathcal{A}
- ▶ a transition from q_1 to q_2 with character c in $\det(\mathcal{A})$ is possible if
 - ▶ in \mathcal{A} there is a transition with c
 - ▶ from **one** of the states that q_1 represents
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- ▶ a state in $\text{det}(\mathcal{A})$ is accepting if it contains an accepting state

To this end, we define three auxiliary functions.

- ▶ ec to compute the ε closure of a state
- ▶ δ^* to compute possible successors of a state
- ▶ $\hat{\delta}$, the new transition function for the generated DFA

Step 1: ϵ closure of an NFA

The ϵ closure of a state q contains all states the NFA can change to by means of ϵ transitions starting from q .

Step 1: ε closure of an NFA

The ε closure of a state q contains all states the NFA can change to by means of ε transitions starting from q .

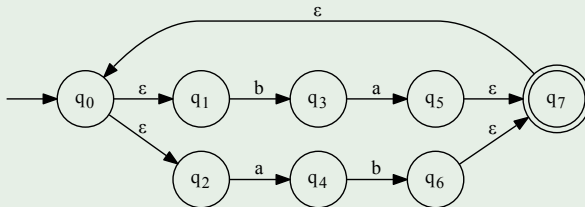
Definition (ε closure)

The function $ec : Q \rightarrow 2^Q$ is the smallest function with the properties:

- ▶ $q \in ec(q)$
- ▶ $p \in ec(q) \wedge (p, \varepsilon, r) \in \Delta \Rightarrow r \in ec(q)$

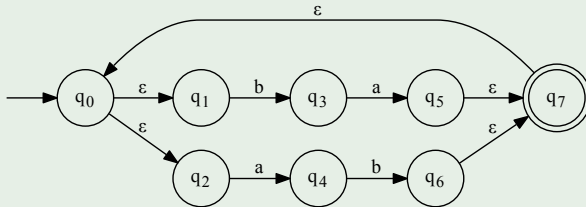
Example: ϵ closure

Example



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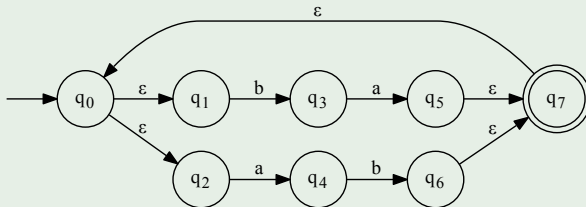
Example



► $ec(q_0) =$

Example: ϵ closure

Example

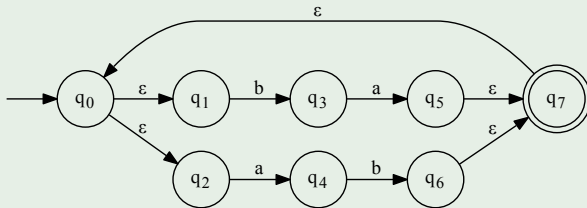


▶ $ec(q_0) = \{q_0, q_1, q_2\}$,

▶ $ec(q_1) =$

Example: ϵ closure

Example



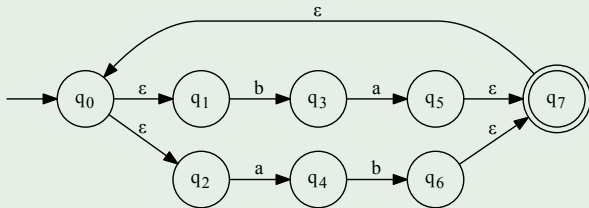
▶ $ec(q_0) = \{q_0, q_1, q_2\}$,

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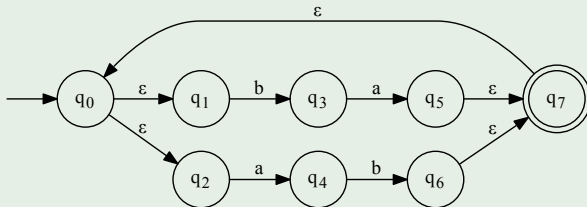
▶ $ec(q_1) = \{q_1\}$,

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▶ $ec(q_3) =$

Example: ϵ closure

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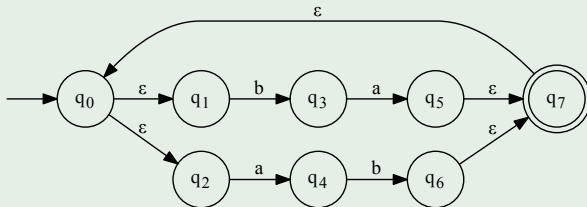
▶ $ec(q_2) = \{q_2\}$,

▶ $ec(q_3) = \{q_3\}$,

▶ $ec(q_4) =$

Example: ϵ closure

Example



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▶ $ec(q_1) = \{q_1\}$,

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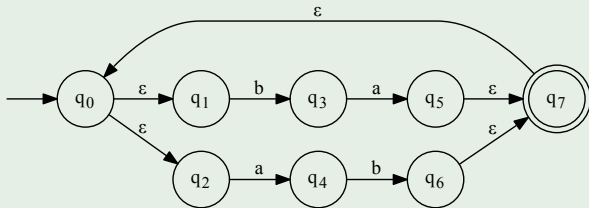
▶ $ec(q_3) = \{q_3\}$,

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▶ $ec(q_5) =$

Example: ϵ closure

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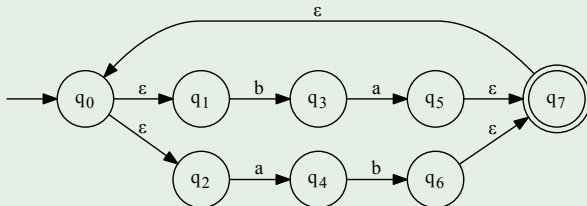
▶ $ec(q_4) = \{q_4\}$,

▶ $ec(q_5) = \{q_5, q_7, q_0, q_1, q_2\}$,

▶ $ec(q_6) =$

Example: ϵ closure

Example



▶ $ec(q_0) = \{q_0, q_1, q_2\}$,

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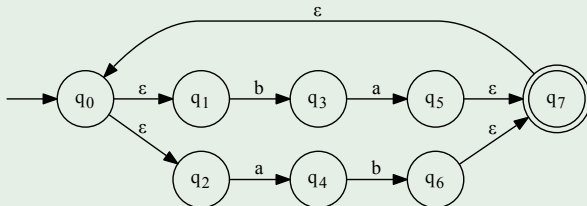
▶ $ec(q_5) = \{q_5, q_7, q_0, q_1, q_2\}$,

▶ $ec(q_6) = \{q_6, q_7, q_0, q_1, q_2\}$,

▶ $ec(q_7) =$

Example: ϵ closure

Example



▶ $ec(q_0) = \{q_0, q_1, q_2\}$,

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Step 2: Successor state function for NFAs

The function δ^* maps

- ▶ a pair (q, c)
- ▶ to the set of **all** states the NFA can change to from q with c
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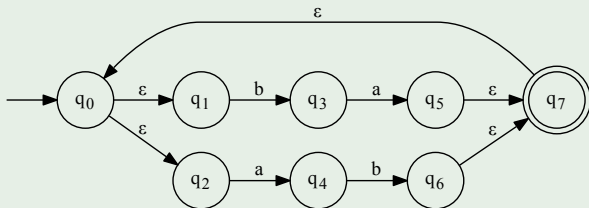
Definition (Successor state function)

The function $\delta^* : Q \times \Sigma \rightarrow 2^Q$ is defined as follows:

$$\delta^*(q, c) = \bigcup_{r \in Q: (q, c, r) \in \Delta} ec(r)$$

Example: successor state function

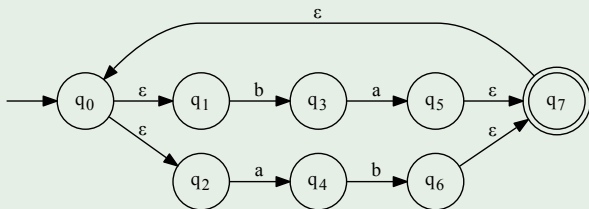
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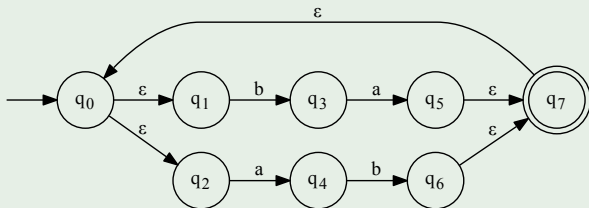


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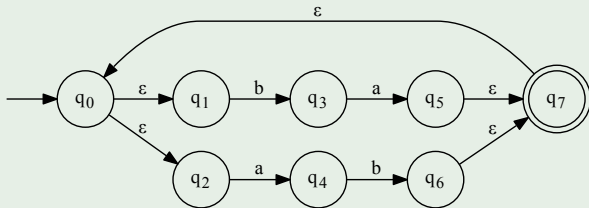
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Example: successor state function

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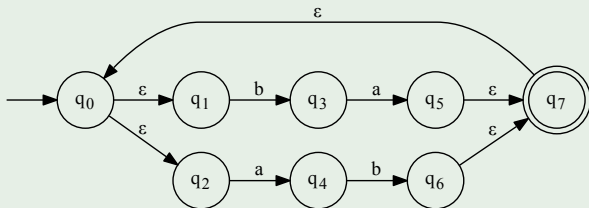


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- ▶ ...

Step 3: extended transition function

The function $\hat{\delta}$ maps

- ▶ a pair (M, c) consisting of a **set** of states M and a character c
- ▶ to the **set** N of states that are reachable from **any** state of M via Δ by reading the character c
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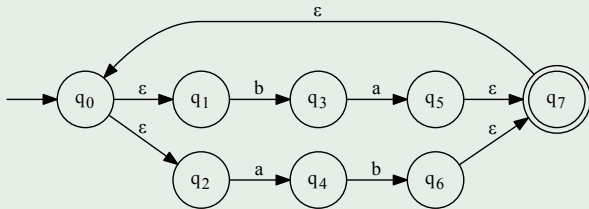
Definition (Extended transition function)

The function $\hat{\delta} : 2^Q \times \Sigma \rightarrow 2^Q$ is defined as follows:

$$\hat{\delta}(M, c) = \bigcup_{q \in M} \delta^*(q, c).$$

Example: extended transition function

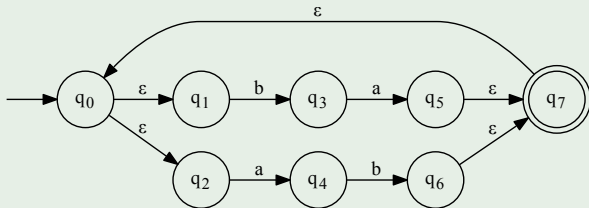
Example



- ▶ $\delta^*(q_0, a) = \{\}$
- ▶ $\delta^*(q_1, b) = \{q_3\}$
- ▶ $\delta^*(q_3, a) = \{q_5, q_7, q_0, q_1, q_2\}$
- ▶ ...

Example: extended transition function

Example



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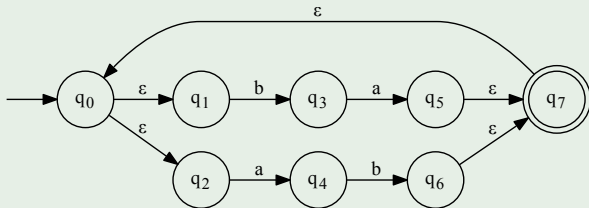
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▶ ...

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Example: extended transition function

Example



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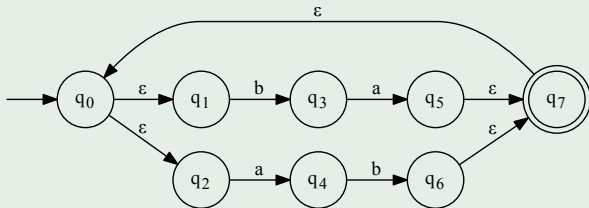
▶ ...

▶ $\hat{\delta}(\{q_0, q_1, q_2\}, a) = \{q_4\}$

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Example: extended transition function

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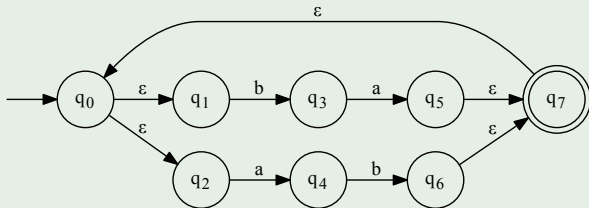
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Equivalence of DFA and NFA: formal definition

Using the auxiliary functions ec , $\hat{\delta}$, we can define $\text{det}(\mathcal{A})$.

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Definition

For an NFA $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$, the **deterministic** Automaton $\det(\mathcal{A})$ is defined as

$$(2^Q, \Sigma, \hat{\delta}, ec(q_0), \hat{F})$$

with $\hat{F} = \{M \in 2^Q \mid M \cap F \neq \{\}\}$.

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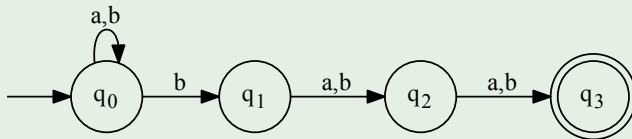
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Remark

In practice, we use a more efficient stepwise construction that only builds the *reachable* states, not all of 2^Q !

Example: transformation into DFA

Example (NFA \mathcal{B} for $(a + b)^*b(a + b)(a + b)$)

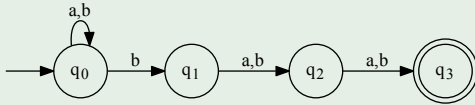


$$\begin{aligned}\mathcal{B} &= (\{q_0, q_1, q_2, q_3\}, \{a, b\}, \Delta, q_0, \{q_3\}) \\ \text{det}(\mathcal{B}) &= (\hat{Q}, \{a, b\}, \hat{\delta}, S_0, \hat{F})\end{aligned}$$

► Initial state: $S_0 := ec(q_0) = \{q_0\}$

Example: transformation into DFA (cont')

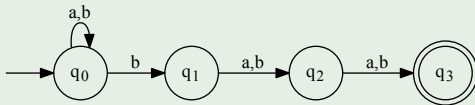
Example



▶ $\hat{\delta}(S_0, a) =$

Example: transformation into DFA (cont')

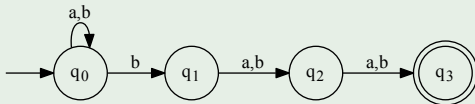
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) =$

Example: transformation into DFA (cont')

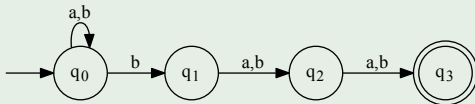
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶ $\hat{\delta}(S_1, a) =$

Example: transformation into DFA (cont')

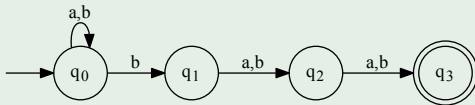
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶ $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶ $\hat{\delta}(S_1, b) =$

Example: transformation into DFA (cont')

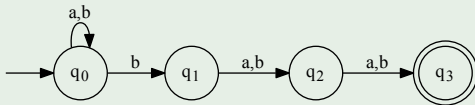
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶ $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶ $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶ $\hat{\delta}(S_2, a) =$

Example: transformation into DFA (cont')

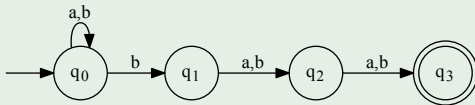
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶ $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶ $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶ $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
- ▶ $\hat{\delta}(S_2, b) =$

Example: transformation into DFA (cont')

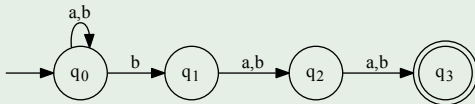
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶ $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶ $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶ $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
- ▶ $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
- ▶ $\hat{\delta}(S_4, a) =$

Example: transformation into DFA (cont')

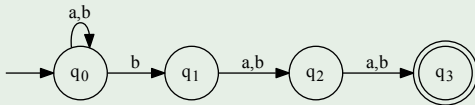
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶ $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶ $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶ $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
- ▶ $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
- ▶ $\hat{\delta}(S_4, a) = \{q_0, q_2, q_3\} =: S_6$
- ▶ $\hat{\delta}(S_4, b) =$

Example: transformation into DFA (cont')

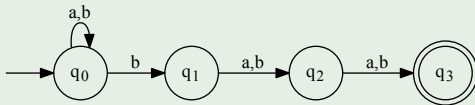
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶ $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶ $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶ $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
- ▶ $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
- ▶ $\hat{\delta}(S_4, a) = \{q_0, q_2, q_3\} =: S_6$
- ▶ $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$

Example: transformation into DFA (cont')

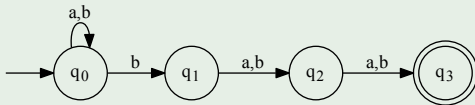
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
 - ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
 - ▶ $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
 - ▶ $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
 - ▶ $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
 - ▶ $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
 - ▶ $\hat{\delta}(S_4, a) = \{q_0, q_2, q_3\} =: S_6$
 - ▶ $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$
- ▶ $\hat{\delta}(S_3, a) =$

Example: transformation into DFA (cont')

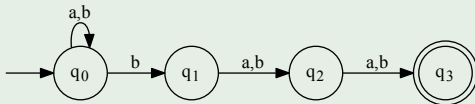
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶ $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶ $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶ $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
- ▶ $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
- ▶ $\hat{\delta}(S_4, a) = \{q_0, q_2, q_3\} =: S_6$
- ▶ $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$
- ▶ $\hat{\delta}(S_3, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_3, b) =$

Example: transformation into DFA (cont')

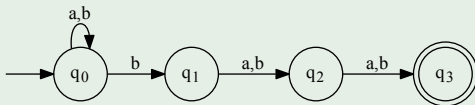
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
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- ▶ $\hat{\delta}(S_3, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_3, b) = \{q_0, q_1\} = S_1$
- ▶ $\hat{\delta}(S_5, a) =$

Example: transformation into DFA (cont')

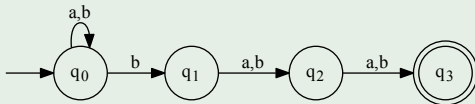
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
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- ▶ $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$
- ▶ $\hat{\delta}(S_3, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_3, b) = \{q_0, q_1\} = S_1$
- ▶ $\hat{\delta}(S_5, a) = \{q_0, q_2\} = S_2$
- ▶ $\hat{\delta}(S_5, b) =$

Example: transformation into DFA (cont')

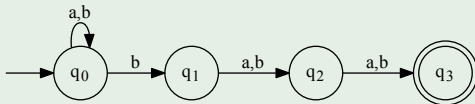
Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
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- ▶ $\hat{\delta}(S_3, a) = \{q_0\} = S_0$
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- ▶ $\hat{\delta}(S_5, b) = \{q_0, q_1, q_2\} = S_4$
- ▶ $\hat{\delta}(S_6, a) =$

Example: transformation into DFA (cont')

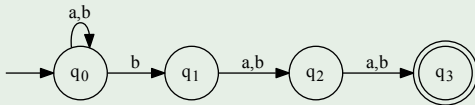
Example



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- ▶ $\hat{\delta}(S_6, b) =$

Example: transformation into DFA (cont')

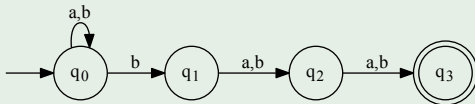
Example



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- ▶ $\hat{\delta}(S_7, a) =$

Example: transformation into DFA (cont')

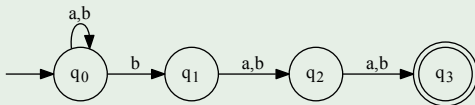
Example



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- ▶ $\hat{\delta}(S_3, a) = \{q_0\} = S_0$
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- ▶ $\hat{\delta}(S_7, a) = \{q_0, q_2, q_3\} = S_6$
- ▶ $\hat{\delta}(S_7, b) =$

Example: transformation into DFA (cont')

Example



- ▶ $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶ $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
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- ▶ $\hat{\delta}(S_6, b) = \{q_0, q_1, q_3\} = S_5$
- ▶ $\hat{\delta}(S_7, a) = \{q_0, q_2, q_3\} = S_6$
- ▶ $\hat{\delta}(S_7, b) = \{q_0, q_1, q_2, q_3\} = S_7$

Example: transformation into DFA (cont')

Example

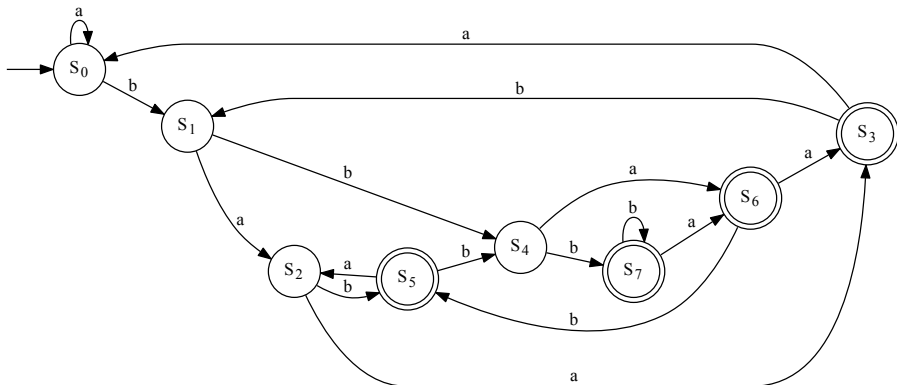
We can now define the DFA $\text{det}(\mathcal{B}) = (\hat{Q}, \Sigma, \hat{\delta}, S_0, \hat{F})$ as follows:

- ▶ the set of states $\hat{Q} = \{S_0, \dots, S_7\}$,
- ▶ the state transition function $\hat{\delta}$ is:

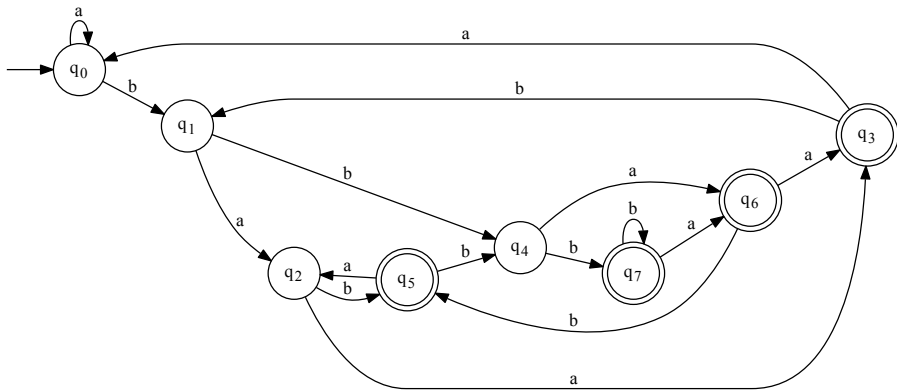
$\hat{\delta}$	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7
a	S_0	S_2	S_3	S_0	S_6	S_2	S_3	S_6
b	S_1	S_4	S_5	S_1	S_7	S_4	S_5	S_7

- ▶ and the set of final states $\hat{F} = \{S_3, S_5, S_6, S_7\}$.

Example: transformation into DFA (cont')

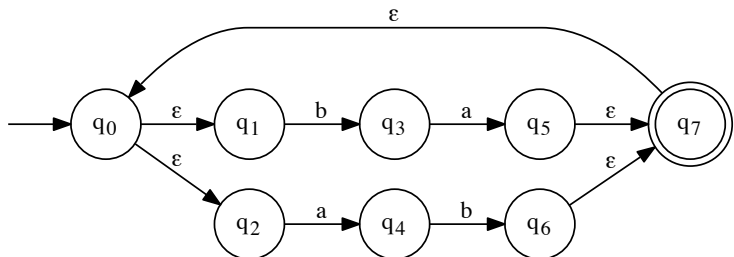


Example: transformation into DFA (cont')



Exercise: Transformation into DFA

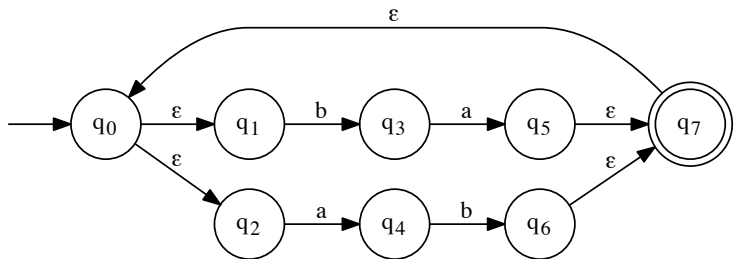
Given the following NFA \mathcal{A} :



- Determine $\det(\mathcal{A})$.
- Draw $\det(\mathcal{A})$'s graphical representation
- Give a regular expression representing the same language as \mathcal{A} .

Exercise: Transformation into DFA

Given the following NFA \mathcal{A} :

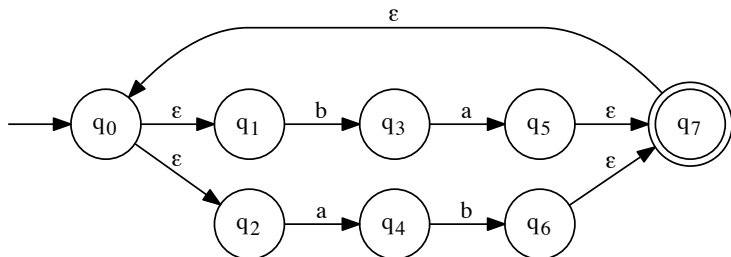


- Determine $\det(\mathcal{A})$.
- Draw $\det(\mathcal{A})$'s graphical representation
- Give a regular expression representing the same language as \mathcal{A} .

Solution

Exercise: Transformation into DFA

Given the following NFA \mathcal{A} :



- Determine $\text{det}(\mathcal{A})$.
- Draw $\text{det}(\mathcal{A})$'s graphical representation
- Give a regular expression representing the same language as \mathcal{A} .

Solution

End lecture 5

Introduction

Regular Languages and Finite Automata

Regular Expressions

Finite Automata

Non-Determinism

Regular expressions and Finite Automata

Minimisation

Equivalence

The Pumping Lemma

Properties of Regular Languages

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

Regular expressions and Finite Automata

- ▶ Regular expressions describe **regular languages**
 - ▶ For each regular language L , there is an regular expression r with $L(r) = L$
 - ▶ For every regular expression r , $L(r)$ is a regular language
- ▶ Finite automata describe **regular languages**
 - ▶ For each regular language L , there is a FA \mathcal{A} with $L(\mathcal{A}) = L$
 - ▶ For every finite automaton \mathcal{A} , $L(\mathcal{A})$ is a regular language
- ▶ Now: constructive proof of equivalence between REs and FAs
 - ▶ We already know that DFAs and NFAs are equivalent
 - ▶ Now: Equivalence of NFAs and REs

Transformation of regular expressions into NFAs

- ▶ For a regular expression r , derive NFA $\mathcal{A}(r)$ with $L(\mathcal{A}(r)) = L(r)$.
- ▶ Idea:
 - ▶ Construct NFAs for the elementary REs ($\emptyset, \varepsilon, c \in \Sigma$)
 - ▶ We combine NFAs for subexpressions to generate NFAs for composite REs
- ▶ All NFAs we construct have a number of special properties:
 - ▶ There are no transitions to the initial state.
 - ▶ There is only a single final state.
 - ▶ There are no transitions from the final state.

Transformation of regular expressions into NFAs

- ▶ For a regular expression r , derive NFA $\mathcal{A}(r)$ with $L(\mathcal{A}(r)) = L(r)$.
- ▶ Idea:
 - ▶ Construct NFAs for the elementary REs ($\emptyset, \varepsilon, c \in \Sigma$)
 - ▶ We combine NFAs for subexpressions to generate NFAs for composite REs
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We can easily achieve this with ε -transitions!

Reminder: Regular Expression

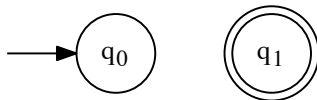
Let Σ be an alphabet.

- ▶ The elementary regular expressions over Σ are:
 - ▶ \emptyset with $L(\emptyset) = \{\}$
 - ▶ ε with $L(\varepsilon) = \{\varepsilon\}$
 - ▶ $c \in \Sigma$ with $L(c) = \{c\}$
- ▶ Let r_1 and r_2 be regular expressions over Σ .
Then the following are also regular expressions over Σ :
 - ▶ $r_1 + r_2$ with $L(r_1 + r_2) = L(r_1) \cup L(r_2)$
 - ▶ $r_1 \cdot r_2$ with $L(r_1 \cdot r_2) = L(r_1) \cdot L(r_2)$
 - ▶ r_1^* with $L(r_1^*) = (L(r_1))^*$

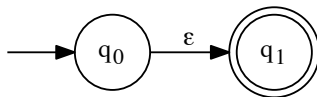
NFAs for elementary REs

Let Σ be the alphabet which r is based on.

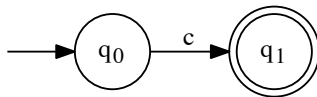
1 $\mathcal{A}(\emptyset) = (\{q_0, q_1\}, \Sigma, \{\}, q_0, \{q_1\})$



2 $\mathcal{A}(\varepsilon) = (\{q_0, q_1\}, \Sigma, \{(q_0, \varepsilon, q_1)\}, q_0, \{q_1\})$



3 $\mathcal{A}(c) = (\{q_0, q_1\}, \Sigma, \{(q_0, c, q_1)\}, q_0, \{q_1\})$ for all $c \in \Sigma$

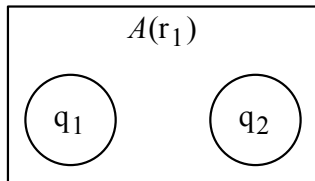


NFAs for composite REs (general)

- ▶ Assume in the following:
 - ▶ $\mathcal{A}(r_1) = (Q_1, \Sigma, \Delta_1, q_1, \{q_2\})$
 - ▶ $\mathcal{A}(r_2) = (Q_2, \Sigma, \Delta_2, q_3, \{q_4\})$
 - ▶ $Q_1 \cap Q_2 = \{\}$
 - ▶ $q_0, q_5 \notin Q_1 \cup Q_2$
- ▶ $\mathcal{A}(r_1)$ is visualised by a square box with two explicit states
 - ▶ The initial state q_1 is on the left
 - ▶ The unique accepting state q_2 on the right
 - ▶ All other states and transitions are implicit
 - ▶ We mark initial/accepting states only for the composite automaton

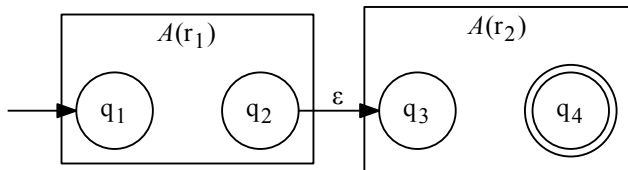
NFAs for composite REs (general)

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 - ▶ All other states and transitions are implicit
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NFAs for composite REs (concatenation)

4 $\mathcal{A}(r_1 \cdot r_2) = (Q_1 \cup Q_2, \Sigma, \Delta_1 \cup \Delta_2 \cup \{(q_2, \varepsilon, q_3)\}, q_1, \{q_4\})$

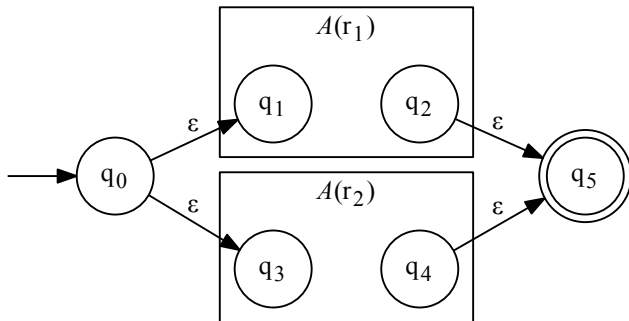


Reminder:

- ▶ $\mathcal{A}(r_1) = (Q_1, \Sigma, \Delta_1, q_1, \{q_2\})$
- ▶ $\mathcal{A}(r_2) = (Q_2, \Sigma, \Delta_2, q_3, \{q_4\})$

NFAs for composite REs (alternatives)

- 5 $\mathcal{A}(r_1 + r_2) = (\{q_0, q_5\} \cup Q_1 \cup Q_2, \Sigma, \Delta, q_0, \{q_5\})$
 $\Delta = \Delta_1 \cup \Delta_2 \cup \{(q_0, \varepsilon, q_1), (q_0, \varepsilon, q_3), (q_2, \varepsilon, q_5), (q_4, \varepsilon, q_5)\}$

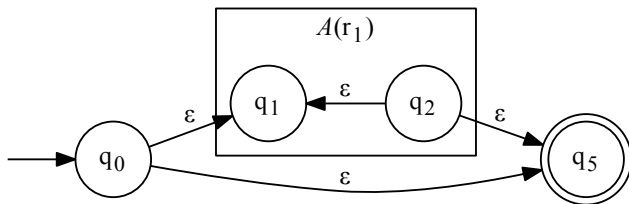


Reminder:

- ▶ $\mathcal{A}(r_1) = (Q_1, \Sigma, \Delta_1, q_1, \{q_2\})$
- ▶ $\mathcal{A}(r_2) = (Q_2, \Sigma, \Delta_2, q_3, \{q_4\})$

NFAs for composite REs (Kleene Star)

- 6 $\mathcal{A}(r_1^*) = (\{q_0, q_5\} \cup Q_1, \Sigma, \Delta, q_0, \{q_5\})$
 $\Delta = \Delta_1 \cup \{(q_0, \varepsilon, q_1), (q_2, \varepsilon, q_1), (q_0, \varepsilon, q_5), (q_2, \varepsilon, q_5)\}$



Reminder:

- $\mathcal{A}(r_1) = (Q_1, \Sigma, \Delta_1, q_1, \{q_2\})$

Result: NFAs can simulate REs

The previous construction produces for each regular expression r an NFA \mathcal{A} with $L(\mathcal{A}) = L(r)$.

Result: NFAs can simulate REs

The previous construction produces for each regular expression r an NFA \mathcal{A} with $L(\mathcal{A}) = L(r)$.

Corollary

Every language described by a regular expression can be accepted by a non-deterministic finite automaton.

Exercise: transformation of RE into NFA

- ▶ Systematically construct an NFA accepting the same language as the regular expression

$$(a + b)a^*b$$

Exercise: transformation of RE into NFA

- ▶ Systematically construct an NFA accepting the same language as the regular expression

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Exercise: transformation of RE into NFA

- ▶ Systematically construct an NFA accepting the same language as the regular expression

$$(a + b)a^*b$$

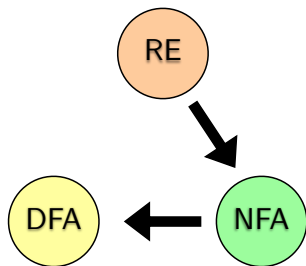
Solution

Overview and orientation

- ▶ Claim: NFAs, DFAs and REs all describe the **same** language class

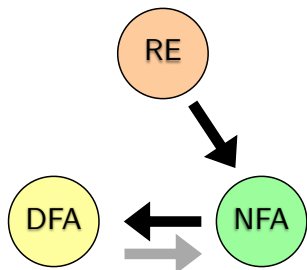
Overview and orientation

- ▶ Claim: NFAs, DFAs and REs all describe the **same** language class
- ▶ Previous transformations:
 - ▶ REs into equivalent NFAs
 - ▶ NFAs into equivalent DFAs



Overview and orientation

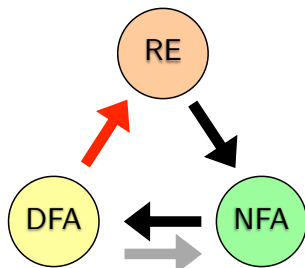
- ▶ Claim: NFAs, DFAs and REs all describe the **same** language class
- ▶ Previous transformations:
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 - ▶ NFAs into equivalent DFAs
 - ▶ (DFAs to equivalent NFAs)



Overview and orientation

- ▶ Claim: NFAs, DFAs and REs all describe the **same** language class
- ▶ Previous transformations:
 - ▶ REs into equivalent NFAs
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 - ▶ (DFAs to equivalent NFAs)

Todo: convert DFA to equivalent RE



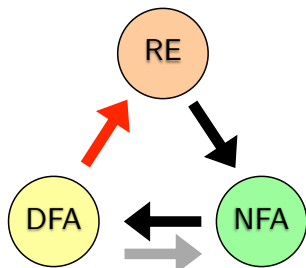
Overview and orientation

- ▶ Claim: NFAs, DFAs and REs all describe the **same** language class
- ▶ Previous transformations:
 - ▶ REs into equivalent NFAs
 - ▶ NFAs into equivalent DFAs
 - ▶ (DFAs to equivalent NFAs)

Todo: convert DFA to equivalent RE

- ▶ Given a DFA \mathcal{A} , derive a regular expression $r(\mathcal{A})$ accepting the same language:

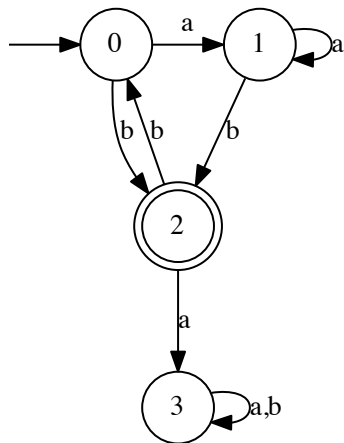
$$L(r(\mathcal{A})) = L(\mathcal{A})$$



Convert DFA into RE

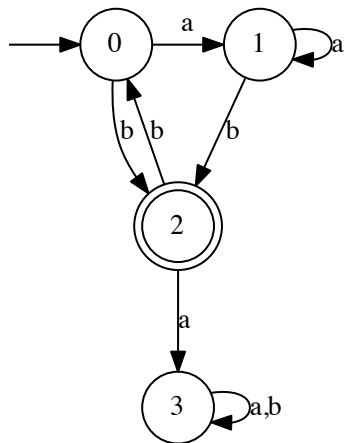
- ▶ Goal: transform DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ into RE $r(\mathcal{A})$ with $L(r(\mathcal{A})) = L(\mathcal{A})$
- ▶ Idea:
 - ▶ For each state q generate an equation describing the language L_q
 - ▶ that is accepted when starting from q ,
 - ▶ depending on the languages accepted at neighbouring states.
 - ▶ Method:
 - ▶ For each transition with c to q' : generate alternative $c \cdot L_{q'}$
 - ▶ For final states: additionally ε
- ▶ Solve the resulting system for L_{q_0}
 - ▶ Result: RE describing $L_{q_0} = L(\mathcal{A})$
- ▶ Convention:
 - ▶ States are named $\{0, 1, \dots, n\}$
 - ▶ Initial state is 0

Convert DFA to RE: Example



► $L_0 \doteq$

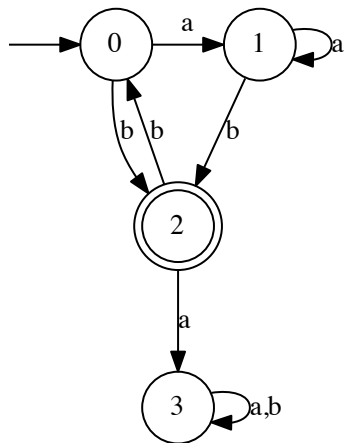
Convert DFA to RE: Example



▶ $L_0 \doteq aL_1 + bL_2$

▶ $L_1 \doteq$

Convert DFA to RE: Example

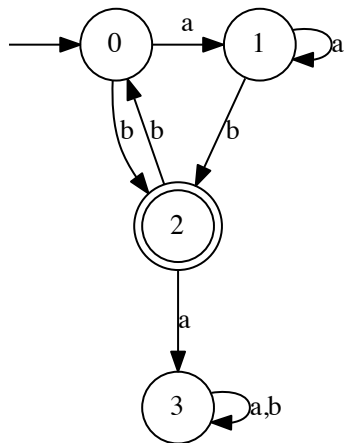


▶ $L_0 \doteq aL_1 + bL_2$

▶ $L_1 \doteq aL_1 + bL_2$

▶ $L_2 \doteq$

Convert DFA to RE: Example



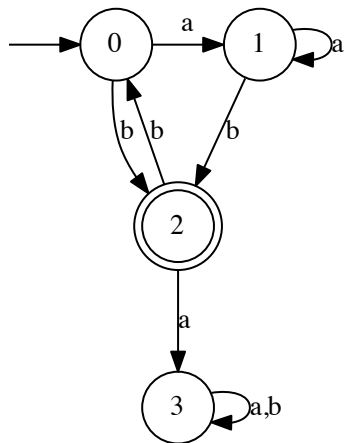
▶ $L_0 \doteq aL_1 + bL_2$

▶ $L_1 \doteq aL_1 + bL_2$

▶ $L_2 \doteq aL_3 + bL_0 + \varepsilon$

▶ $L_3 \doteq$

Convert DFA to RE: Example



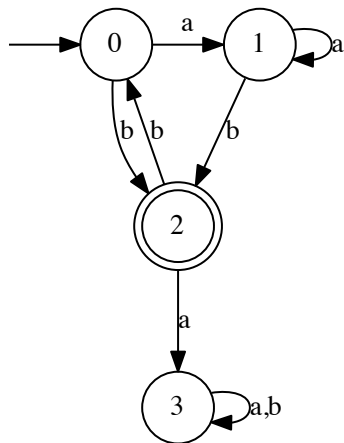
▶ $L_0 \doteq aL_1 + bL_2$

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Convert DFA to RE: Example



▶ $L_0 \doteq aL_1 + bL_2$

▶ $L_1 \doteq aL_1 + bL_2$

▶ $L_2 \doteq aL_3 + bL_0 + \varepsilon$

▶ $L_3 \doteq (a + b)L_3$

4 equations, 4 unknowns

What now?

Insert: Arden's Lemma

Lemma:

$$\varepsilon \notin L(s) \text{ and } r \doteq sr + t \longrightarrow r \doteq s^*t$$

Arden, Dean N.:
*Delayed-logic
and finite-state
machines*,
Proceedings of
the Second
Annual
Symposium on
Switching
Circuit Theory
and Logical
Design, 1961,
pp. 133–151,
IEEE

Insert: Arden's Lemma

Lemma:

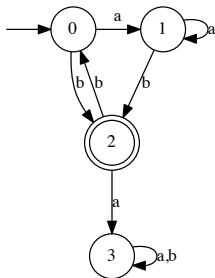
$$\varepsilon \notin L(s) \text{ and } r \doteq sr + t \longrightarrow r \doteq s^*t$$

Compare Arto Salomaa:

$$\varepsilon \notin L(s) \text{ and } r \doteq rs + t \longrightarrow r \doteq ts^*$$

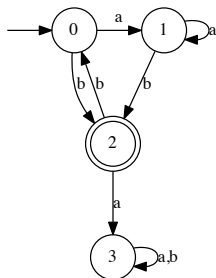
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IEEE

Convert DFA to RE: Example



- ▶ $L_0 \doteq aL_1 + bL_2$
- ▶ $L_1 \doteq aL_1 + bL_2$
- ▶ $L_2 \doteq aL_3 + bL_0 + \varepsilon$
- ▶ $L_3 \doteq (a + b)L_3$

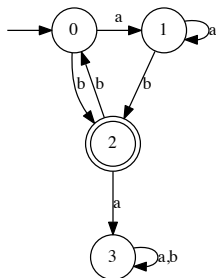
Convert DFA to RE: Example



- ▶ $L_0 \doteq aL_1 + bL_2$
- ▶ $L_1 \doteq aL_1 + bL_2$
- ▶ $L_2 \doteq aL_3 + bL_0 + \varepsilon$
- ▶ $L_3 \doteq (a + b)L_3$

$$L_3 \doteq (a + b)L_3 + \emptyset \quad \text{[neutral el.]}$$

Convert DFA to RE: Example



▶ $L_0 \doteq aL_1 + bL_2$

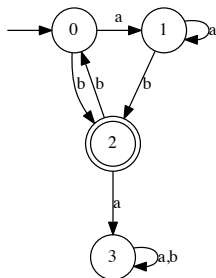
▶ $L_1 \doteq aL_1 + bL_2$

▶ $L_2 \doteq aL_3 + bL_0 + \varepsilon$

▶ $L_3 \doteq (a + b)L_3$

$$\begin{aligned} L_3 &\doteq (a + b)L_3 + \emptyset && \text{[neutral el.]} \\ &\doteq (a + b)^*\emptyset && \text{[Arden]} \end{aligned}$$

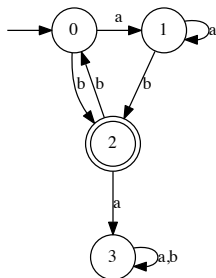
Convert DFA to RE: Example



- ▶ $L_0 \doteq aL_1 + bL_2$
- ▶ $L_1 \doteq aL_1 + bL_2$
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$$\begin{aligned} L_3 &\doteq (a + b)L_3 + \emptyset && \text{[neutral el.]} \\ &\doteq (a + b)^*\emptyset && \text{[Arden]} \\ &\doteq \emptyset && \text{[absorbing el.]} \end{aligned}$$

Convert DFA to RE: Example



$$\blacktriangleright L_0 \doteq aL_1 + bL_2$$

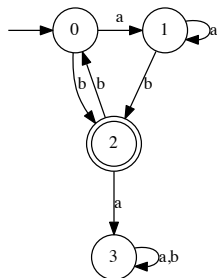
$$\blacktriangleright L_1 \doteq aL_1 + bL_2$$

$$\blacktriangleright L_2 \doteq aL_3 + bL_0 + \varepsilon$$

$$\blacktriangleright L_3 \doteq (a + b)L_3$$

$$\begin{aligned} L_3 &\doteq (a + b)L_3 + \emptyset && \text{[neutral el.]} \\ &\doteq (a + b)^*\emptyset && \text{[Arden]} \\ &\doteq \emptyset && \text{[absorbing el.]} \\ L_2 &\doteq a\emptyset + bL_0 + \varepsilon && \text{[replace } L_3\text{]} \end{aligned}$$

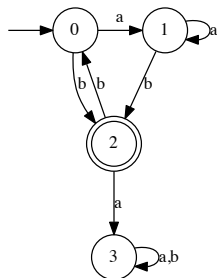
Convert DFA to RE: Example



- ▶ $L_0 \doteq aL_1 + bL_2$
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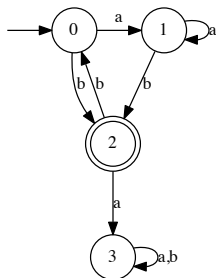
Convert DFA to RE: Example



- ▶ $L_0 \doteq aL_1 + bL_2$
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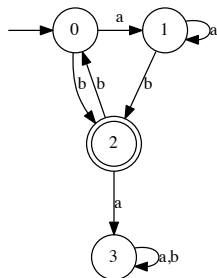
Convert DFA to RE: Example



- ▶ $L_0 \doteq aL_1 + bL_2$
- ▶ $L_1 \doteq aL_1 + bL_2$
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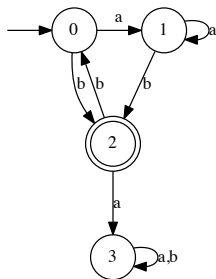
Convert DFA to RE: Example



- ▶ $L_0 \doteq aL_1 + bL_2$
- ▶ $L_1 \doteq aL_1 + bL_2$
- ▶ $L_2 \doteq aL_3 + bL_0 + \varepsilon$
- ▶ $L_3 \doteq (a + b)L_3$

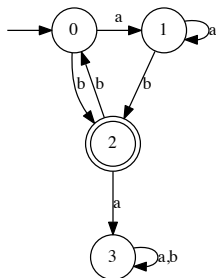
$$\begin{aligned} L_3 &\doteq (a + b)L_3 + \emptyset && \text{[neutral el.]} \\ &\doteq (a + b)^*\emptyset && \text{[Arden]} \\ &\doteq \emptyset && \text{[absorbing el.]} \\ L_2 &\doteq a\emptyset + bL_0 + \varepsilon && \text{[replace } L_3\text{]} \\ &\doteq \emptyset + bL_0 + \varepsilon && \text{[absorbing el.]} \\ &\doteq bL_0 + \varepsilon && \text{[neutral el.]} \\ L_1 &\doteq aL_1 + b(bL_0 + \varepsilon) && \text{[replace } L_2\text{]} \\ &\doteq a^*b(bL_0 + \varepsilon) && \text{[Arden]} \end{aligned}$$

Convert DFA to RE: Example (continued)



- ▶ $L_0 \doteq aL_1 + bL_2$
- ▶ $L_1 \doteq a^*b(bL_0 + \varepsilon)$
- ▶ $L_2 \doteq bL_0 + \varepsilon$
- ▶ $L_3 \doteq \emptyset$

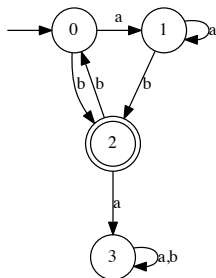
Convert DFA to RE: Example (continued)



- ▶ $L_0 \doteq aL_1 + bL_2$
- ▶ $L_1 \doteq a^*b(bL_0 + \varepsilon)$
- ▶ $L_2 \doteq bL_0 + \varepsilon$
- ▶ $L_3 \doteq \emptyset$

$$L_0 \doteq a(a^*b(bL_0 + \varepsilon)) + b(bL_0 + \varepsilon) \quad \text{[replace } L_1, L_2]$$

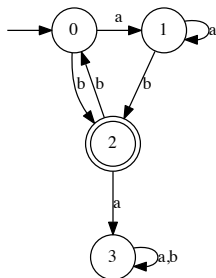
Convert DFA to RE: Example (continued)



- ▶ $L_0 \doteq aL_1 + bL_2$
- ▶ $L_1 \doteq a^*b(bL_0 + \varepsilon)$
- ▶ $L_2 \doteq bL_0 + \varepsilon$
- ▶ $L_3 \doteq \emptyset$

$$\begin{aligned} L_0 &\doteq a(a^*b(bL_0 + \varepsilon)) + b(bL_0 + \varepsilon) && \text{[replace } L_1, L_2\text{]} \\ &\doteq aa^*bbL_0 + aa^*b + bbL_0 + b && \text{[dist.]} \end{aligned}$$

Convert DFA to RE: Example (continued)



▶ $L_0 \doteq aL_1 + bL_2$

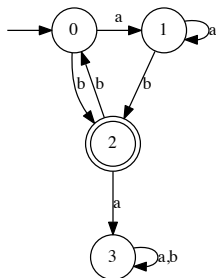
▶ $L_1 \doteq a^*b(bL_0 + \varepsilon)$

▶ $L_2 \doteq bL_0 + \varepsilon$

▶ $L_3 \doteq \emptyset$

$$\begin{aligned} L_0 &\doteq a(a^*b(bL_0 + \varepsilon)) + b(bL_0 + \varepsilon) && \text{[replace } L_1, L_2\text{]} \\ &\doteq aa^*bbL_0 + aa^*b + bbL_0 + b && \text{[dist.]} \\ &\doteq (aa^*bb + bb)L_0 + aa^*b + b && \text{[comm.,dist.]} \end{aligned}$$

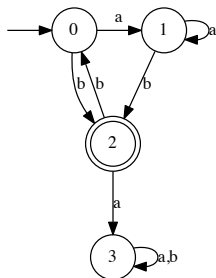
Convert DFA to RE: Example (continued)



- ▶ $L_0 \doteq aL_1 + bL_2$
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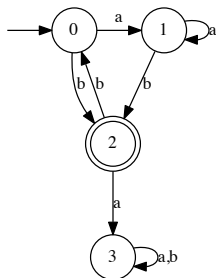
Convert DFA to RE: Example (continued)



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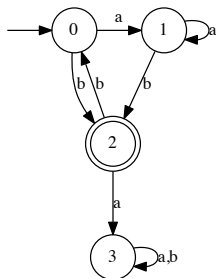
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Convert DFA to RE: Example (continued)



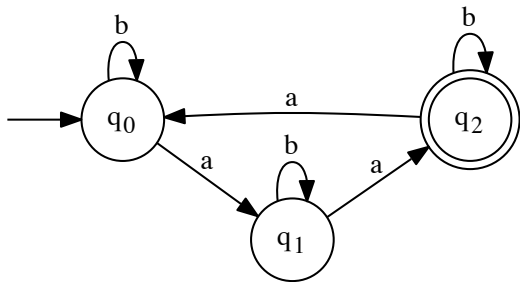
- ▶ $L_0 \doteq aL_1 + bL_2$
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$$\begin{aligned} L_0 &\doteq a(a^*b(bL_0 + \varepsilon)) + b(bL_0 + \varepsilon) && \text{[replace } L_1, L_2\text{]} \\ &\doteq aa^*bbL_0 + aa^*b + bbL_0 + b && \text{[dist.]} \\ &\doteq (aa^*bb + bb)L_0 + aa^*b + b && \text{[comm., dist.]} \\ &\doteq (aa^*bb + bb)^*(aa^*b + b) && \text{[Arden]} \\ &\doteq ((aa^* + \varepsilon)bb)^*((aa^* + \varepsilon)b) && \text{[dist.]} \\ &\doteq (a^*bb)^*(a^*b) && \text{[} rr^* + \varepsilon \doteq r^*\text{]} \end{aligned}$$

Therefore: $L(\mathcal{A}) = L((a^*bb)^*(a^*b))$

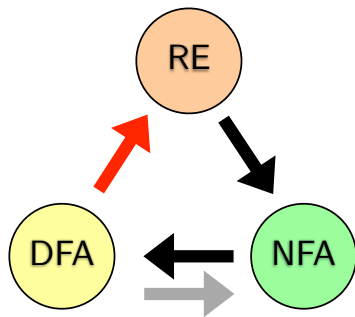
Exercise: conversion from DFA to RE

Transform the following DFA into a regular expression accepting the same language:



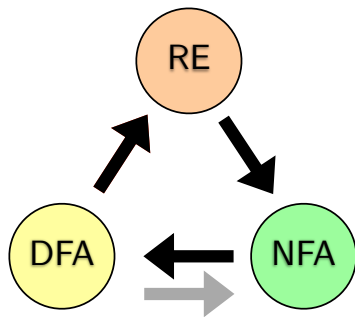
Resume: Finite automata and regular expressions

- ▶ We have learned how to convert
 - ▶ REs to equivalent NFAs
 - ▶ NFAs to equivalent DFAs
 - ▶ (DFAs to equivalent NFAs)



Resume: Finite automata and regular expressions

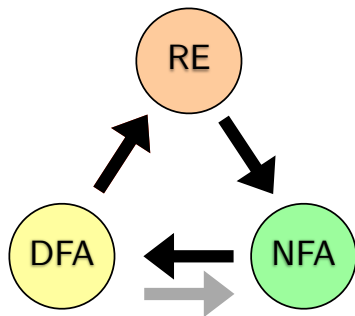
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 - ▶ (DFAs to equivalent NFAs)
 - ▶ **DFAs to equivalent REs**



Resume: Finite automata and regular expressions

- ▶ We have learned how to convert
 - ▶ REs to equivalent NFAs
 - ▶ NFAs to equivalent DFAs
 - ▶ (DFAs to equivalent NFAs)
 - ▶ **DFAs to equivalent REs**

REs, NFAs and DFAs describe the same class of languages – **regular languages!**



**and now it's time for something
completely different**



**and now it's time for something
completely different**



End lecture 6

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Non-Determinism

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Efficient Automata: Minimisation of DFAs

Given the DFA

$$\mathcal{A} = (Q, \Sigma, \delta, q_0, F),$$

we want to derive a DFA

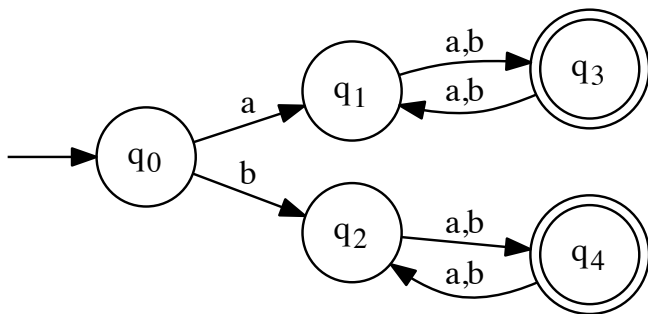
$$\mathcal{A}^- = (Q^-, \Sigma, \delta^-, q_0, F^-),$$

accepting the same language:

$$L(\mathcal{A}) = L(\mathcal{A}^-)$$

for which the **number of states** (elements of Q^-) is **minimal**, i.e. there is no DFA accepting $L(\mathcal{A})$ with fewer states.

Minimisation of DFAs: example/exercise



How small can we make it?

Minimisation of DFAs

Idea: For a DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, identify **pairs of necessarily distinct states**

- ▶ Base case: Two states p, q are necessarily distinct if:
 - ▶ one of them is accepting, the other is not accepting
- ▶ Inductive case: Two states p, q are necessarily distinct if
 - ▶ there is a $c \in \Sigma$ such that $\delta(p, c) = p', \delta(q, c) = q'$
 - ▶ and p', q' are already necessarily distinct

Minimisation of DFAs

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 - ▶ and p', q' are already necessarily distinct

Definition (Necessarily distinct states)

For a DFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$, V is the smallest set of pairs with

- ▶ $\{(p, q) \in (Q \times Q) \mid p \in F, q \notin F\} \subseteq V$
- ▶ $\{(p, q) \in (Q \times Q) \mid p \notin F, q \in F\} \subseteq V$
- ▶ if $\delta(p, c) = p', \delta(q, c) = q', (p', q') \in V$ for some $c \in \Sigma$, then $(p, q) \in V$.

Minimisation of DFAs

- 1 Initialize V with all those pairs for which one member is a final state and the other is not:

$$V = \{(p, q) \in Q \times Q \mid (p \in F \wedge q \notin F) \vee (p \notin F \wedge q \in F)\}.$$

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$$V = \{(p, q) \in Q \times Q \mid (p \in F \wedge q \notin F) \vee (p \notin F \wedge q \in F)\}.$$

- 2 While there exists
 - ▶ a new pair of states (p, q) and a symbol c
 - ▶ such that the states $\delta(p, c)$ and $\delta(q, c)$ are necessarily distinct,
 - ▶ add this pair and its inverse to V :

Minimisation of DFAs

- 1 Initialize V with all those pairs for which one member is a final state and the other is not:

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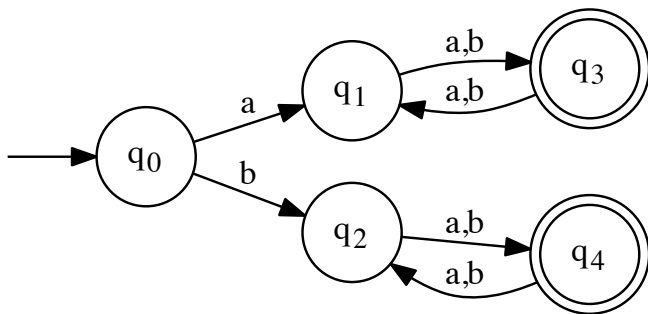
- ▶ a new pair of states (p, q) and a symbol c
- ▶ such that the states $\delta(p, c)$ and $\delta(q, c)$ are necessarily distinct,
- ▶ add this pair and its inverse to V :

```
while ( $\exists(p, q) \in Q \times Q \exists c \in \Sigma \mid (\delta(p, c), \delta(q, c)) \in V \wedge (p, q) \notin V$ )
{
   $V = V \cup \{(p, q), (q, p)\}$ 
}
```


Minimisation of DFAs: merging States

Minimisation of DFAs: example

We want to minimize this DFA with 5 states:



Minimisation of DFAs: example (cont.)

This is the formal definition of the DFA:

$$\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$$

with

1 $Q = \{q_0, q_1, q_2, q_3, q_4\}$

2 $\Sigma = \{a, b.\}$

3 $\delta = \dots$ (skipped to save space, see graph)

4 $F = \{q_3, q_4\}$

Minimisation of DFAs: example (cont.)

This is the formal definition of the DFA:

$$\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$$

with

- 1 $Q = \{q_0, q_1, q_2, q_3, q_4\}$
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- 3 $\delta = \dots$ (skipped to save space, see graph)
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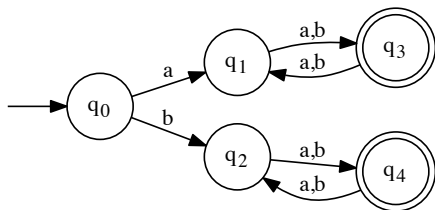
Represent the set V by means of a two-dimensional table with

- ▶ the elements of Q as columns and rows
- ▶ the elements of V are marked with \times
- ▶ pairs that are definitely **not** members of V are marked with \circ

Minimisation of DFAs: example (cont.)

- 1** the initial state of V is obtained by using $F = \{q_3, q_4\}$ and $Q \setminus F = \{q_0, q_1, q_2\}$:

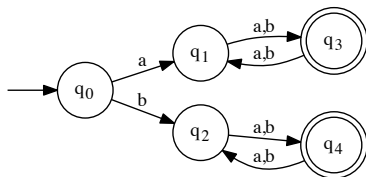
	q_0	q_1	q_2	q_3	q_4
q_0				×	×
q_1				×	×
q_2				×	×
q_3	×	×	×		
q_4	×	×	×		



Minimisation of DFAs: example (cont.)

- 2 The elements of $\{(q_i, q_i) \mid i \in \{0, \dots, 4\}\}$ are not contained in V since every state is indistinguishable from itself:

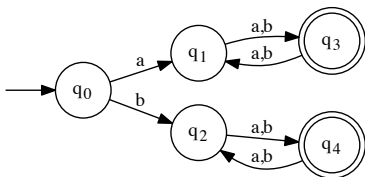
	q_0	q_1	q_2	q_3	q_4
q_0	○			×	×
q_1		○		×	×
q_2			○	×	×
q_3	×	×	×	○	
q_4	×	×	×		○



Minimisation of DFAs: example (cont.)

- 2 The elements of $\{(q_i, q_i) \mid i \in \{0, \dots, 4\}\}$ are not contained in V since every state is indistinguishable from itself:

	q_0	q_1	q_2	q_3	q_4
q_0	○			×	×
q_1		○		×	×
q_2			○	×	×
q_3	×	×	×	○	
q_4	×	×	×		○

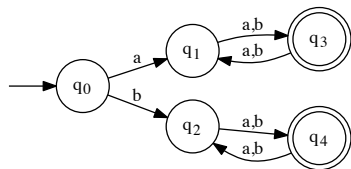


There are eight remaining empty fields. Since the table is symmetric, **four** pairs of states have to be checked.

Minimisation of DFAs: example (cont.)

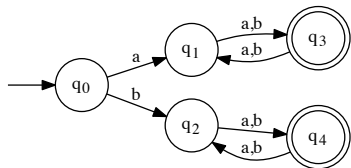
Minimisation of DFAs: example (cont.)

- 3 Check the transitions of **every remaining state-pair** for **every letter**.



Minimisation of DFAs: example (cont.)

- 3** Check the transitions of **every remaining state-pair** for **every letter**.



- 1** $\delta(q_0, a) = q_1; \delta(q_1, a) = q_3; (q_1, q_3) \in V \rightarrow (q_0, q_1), (q_1, q_0) \in V$
- 2** $\delta(q_0, a) = q_1; \delta(q_2, a) = q_4; (q_1, q_4) \in V \rightarrow (q_0, q_2), (q_2, q_0) \in V$
- 3** $\delta(q_1, a) = q_3; \delta(q_2, a) = q_4; (q_3, q_4) \notin V$ (as of yet)
 $\delta(q_1, b) = q_3; \delta(q_2, b) = q_4; (q_3, q_4) \notin V$ (as of yet)
- 4** $\delta(q_3, a) = q_1; \delta(q_4, a) = q_2; (q_1, q_2) \notin V$ (as of yet)
 $\delta(q_3, b) = q_1; \delta(q_4, b) = q_2; (q_1, q_2) \notin V$ (as of yet)

Minimisation of DFAs: example (cont.)

- 4 Mark the newly found distinguishable pairs with \times :

	q_0	q_1	q_2	q_3	q_4
q_0	○	×	×	×	×
q_1	×	○		×	×
q_2	×		○	×	×
q_3	×	×	×	○	
q_4	×	×	×		○

Minimisation of DFAs: example (cont.)

- 4 Mark the newly found distinguishable pairs with \times :

	q_0	q_1	q_2	q_3	q_4
q_0	○	×	×	×	×
q_1	×	○		×	×
q_2	×		○	×	×
q_3	×	×	×	○	
q_4	×	×	×		○

Two pairs remain to be checked.

Minimisation of DFAs: example (cont.)

- 5 Check the remaining pairs.
- 6 Since no additional distinguishable state pairs are found, fill empty cells with \circ :

	q_0	q_1	q_2	q_3	q_4
q_0	\circ	\times	\times	\times	\times
q_1	\times	\circ	\circ	\times	\times
q_2	\times	\circ	\circ	\times	\times
q_3	\times	\times	\times	\circ	\circ
q_4	\times	\times	\times	\circ	\circ

Minimisation of DFAs: example (cont.)

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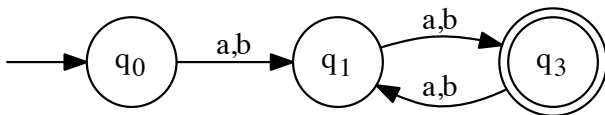
	q_0	q_1	q_2	q_3	q_4
q_0	\circ	\times	\times	\times	\times
q_1	\times	\circ	\circ	\times	\times
q_2	\times	\circ	\circ	\times	\times
q_3	\times	\times	\times	\circ	\circ
q_4	\times	\times	\times	\circ	\circ

From the table, we can derive the following indistinguishable state pairs (omitting trivial and symmetric ones):

- ▶ (q_1, q_2) ,
- ▶ (q_3, q_4) .

Minimisation of DFAs: example (cont.)

- ▶ This is the minimized DFA after merging indistinguishable states:



Minimisation of DFAs: exercise

Derive a minimal DFA accepting the language

$$L(a(ba)^*).$$

Solve the exercise in three steps:

- 1 Derive an NFA accepting L .
- 2 Transform the NFA into a DFA.
- 3 Minimize the DFA.

Uniqueness of minimal DFA

Theorem (The minimal DFA is unique)

*Assume an arbitrary regular language L . Then there is a unique (up to the the renaming of states) **minimal** DFA \mathcal{A} with $L(\mathcal{A}) = L$.*

- ▶ States can easily be systematically renamed to make equivalent minimal automata strictly equal
- ▶ The unique minimal DFA for L can be constructed by minimizing an arbitrary DFA that accepts L

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Equivalence of regular expressions

- ▶ Different regular expressions can describe the **same language**
- ▶ **Algebraic transformation rules** can be used to prove equivalence
 - ▶ requires human interaction
 - ▶ can be very difficult
 - ▶ non-equivalence cannot be shown
- ▶ Now: straight-forward algorithm proving equivalence of REs based on FA
- ▶ The algorithm is described in the textbook by John E. Hopcroft, Rajeev Motwani, Jeffrey D. Ullman: *Introduction to Automata Theory, Languages, and Computation (3rd edition)*, 2007 (and earlier editions)

Equivalence of regular expressions: algorithm

- 1 Given the REs r_1 and r_2 , derive NFAs \mathcal{A}_1 and \mathcal{A}_2 accepting their respective languages:

$$L(r_1) = L(\mathcal{A}_1) \quad \text{and} \quad L(r_2) = L(\mathcal{A}_2).$$

- 2 Transform the NFAs \mathcal{A}_1 and \mathcal{A}_2 into the DFAs \mathcal{D}_1 and \mathcal{D}_2 .
- 3 Minimize the DFAs \mathcal{D}_1 and \mathcal{D}_2 yielding the DFAs \mathcal{M}_1 and \mathcal{M}_2 .
- 4 $r_1 \doteq r_2$ holds iff \mathcal{M}_1 and \mathcal{M}_2 are identical (modulo renaming of states)

Equivalence of regular expressions: algorithm

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- 4 $r_1 \doteq r_2$ holds iff \mathcal{M}_1 and \mathcal{M}_2 are identical (modulo renaming of states)

Note: If equivalence can be shown in any intermediate stage of the algorithm, this is sufficient to prove $r_1 \doteq r_2$ (e.g. if $\mathcal{A}_1 = \mathcal{A}_2$).

Exercise: Equivalence of regular expressions

Reusing an exercise from an earlier section, prove the following equivalence (by conversion to minimal DFAs):

$$10(10)^* \doteq 1(01)^*0$$

Exercise: Equivalence of regular expressions

Reusing an exercise from an earlier section, prove the following equivalence (by conversion to minimal DFAs):

$$10(10)^* \doteq 1(01)^*0$$

Solution

End lecture 7

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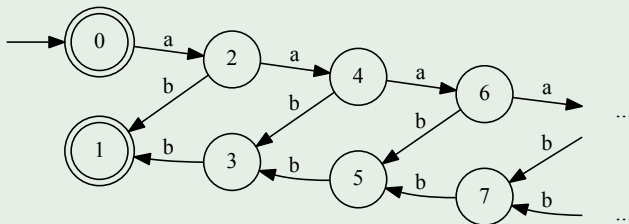
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Non-regular languages

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Example (Naive automaton \mathcal{A} for $L = \{a^n b^n \mid n \in \mathbb{N}\}$)

\mathcal{A} has an infinite number of states:

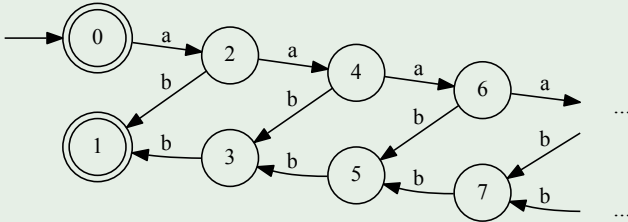


Non-regular languages

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Example (Naive automaton \mathcal{A} for $L = \{a^n b^n \mid n \in \mathbb{N}\}$)

\mathcal{A} has an infinite number of states:



- ▶ Is there a better solution?
- ▶ If no, how can this be shown?

Pumping Lemma: Idea

- 1 Every regular language L is accepted by a **finite** Automaton \mathcal{A}_L .
- 2 If L contains arbitrarily long words, then \mathcal{A}_L must contain a **cycle**.
 - ▶ L contains arbitrarily long words iff L is infinite.
- 3 If \mathcal{A}_L contains a cycle, then the cycle can be traversed **arbitrarily often** (and the resulting word will be accepted).

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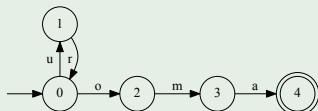


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Example (Cyclic Automaton \mathcal{C})

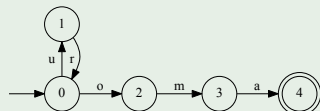


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Example (Cyclic Automaton \mathcal{C})



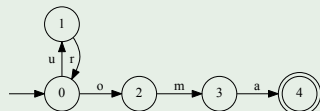
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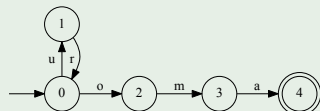
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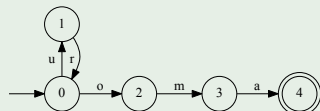
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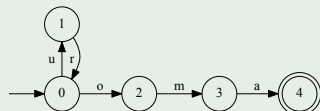
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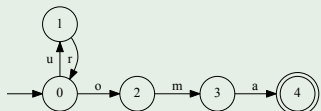
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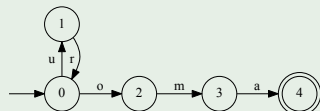
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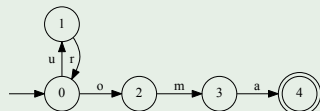
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The Pumping Lemma

Lemma

Let L be a regular language.

Then there exists a $k \in \mathbb{N}$ such that for every word $s \in L$ with $|s| \geq k$ the following holds:

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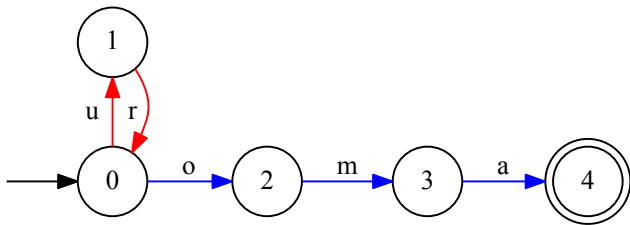
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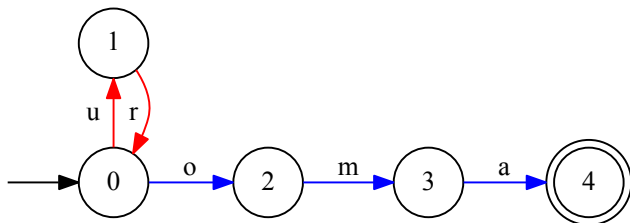
Then there exists a $k \in \mathbb{N}$ such that for every word $s \in L$ with $|s| \geq k$ the following holds:

- 1** $\exists u, v, w \in \Sigma^* (s = u \cdot v \cdot w)$,
*i.e. s consists of **prolog** u , **cycle** v and **epilog** w ,*
- 2** $v \neq \varepsilon$,
*i.e. the cycle has a **length of at least 1**,*
- 3** $|u \cdot v| \leq k$,
*i.e. prolog and cycle combined have a **length of at most k** ,*
- 4** $\forall h \in \mathbb{N} (u \cdot v^h \cdot w \in L)$,
*i.e. an **arbitrary number of cycle transitions** results in a word of the language L .*

The Pumping Lemma visualised



The Pumping Lemma visualised



- ▶ \mathcal{C} has 5 states $k = 5$
- ▶ $uroma$ has 5 letters $s = uroma$
- ▶ There is a segmentation $s = u \cdot v \cdot w$ $u = \varepsilon$ $v = ur$ $w = oma$
- ▶ such that $|v| \neq \varepsilon$ $v = ur$
- ▶ and $|u \cdot v| \leq k$ $|\varepsilon \cdot ur| = 2 \leq 5$
- ▶ and $\forall h \in \mathbb{N}(u \cdot v^h \cdot w \in L(\mathcal{C}))$ $(ur)^*oma \subseteq L(\mathcal{C})$

Using the Pumping Lemma

- ▶ The Pumping Lemma describes a property of **regular** languages
 - ▶ *If L is regular, then some words can be pumped up.*
- ▶ Goal: proof of **irregularity** of a language
 - ▶ *If L has property X , then L is not regular.*
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Theorem (Contraposition)

$$A \rightarrow B \quad \Leftrightarrow \quad \neg B \rightarrow \neg A$$

Contraposition of the Pumping Lemma

The Pumping Lemma in formal logic:

$$\begin{aligned} \text{reg}(L) \rightarrow & \exists k \in \mathbb{N} \forall s \in L : (|s| \geq k \rightarrow \\ & \exists u, v, w : (s = u \cdot v \cdot w \wedge v \neq \varepsilon \wedge |u \cdot v| \leq k \wedge \\ & \forall h \in \mathbb{N} : (u \cdot v^h \cdot w \in L))) \end{aligned}$$

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$$\neg(\exists k \in \mathbb{N} \forall s \in L (|s| \geq k \rightarrow \\ \exists u, v, w (s = u \cdot v \cdot w \wedge v \neq \varepsilon \wedge |u \cdot v| \leq k \wedge \\ \forall h \in \mathbb{N} (u \cdot v^h \cdot w \in L)))) \rightarrow \neg \text{reg}(L)$$

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After pushing negation inward and doing some propositional transformations:

$$\begin{aligned} \forall k \in \mathbb{N} \exists s \in L(|s| \geq k \wedge \\ \forall u, v, w(s = u \cdot v \cdot w \wedge v \neq \varepsilon \wedge |u \cdot v| \leq k \rightarrow \\ \exists h \in \mathbb{N}(u \cdot v^h \cdot w \notin L))) \rightarrow \neg \text{reg}(L) \end{aligned}$$

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then L is not regular.

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We have to show:

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Example ($L = a^n b^n$)

- ▶ Choose $s = a^k b^k$. It follows:

$$s = \underbrace{a^i}_u \cdot \underbrace{a^j}_v \cdot \underbrace{a^\ell \cdot b^k}_w$$

- ▶ $i + j + \ell = k$
- ▶ since $|u \cdot v| \leq k$ holds, u and v consist only of a 's
- ▶ $v \neq \varepsilon$ implies $j \geq 1$

- ▶ Choose $h = 0$. It follows:

- ▶ $u \cdot v^h \cdot w = u \cdot w = a^{i+\ell} b^k$
- ▶ $j \geq 1$ implies $i + \ell < k$
- ▶ $a^{i+\ell} b^k \notin L$

Regarding quantifiers

Four quantifiers:

- ▶ In the lemma:

$$\exists k \forall s \exists u, v, w \forall h (u \cdot v^h \cdot w \in L)$$

- ▶ To show irregularity:

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To do:

- 1 Find a word s depending on the length k .
- 2 Find an h depending on the segmentation $u \cdot v \cdot w$.
- 3 Prove that $u \cdot v^h \cdot w \notin L$ holds.

Exercise: The Pumping Game

Play the pumping game at <http://weitz.de/pump/>.

Exercise: $a^n b^m$ with $n < m$

Use the pumping lemma to show that

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Solution

Challenging exercise / homework

Let L be the number containing all words of the form a^p where p is a prime number:

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Prove that L is not a regular language.

Hint: let $h = p + 1$

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Finite automata cannot **count** arbitrarily high.

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Examples (Nested dependencies)

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```
Erinnern Sie sich,  
    wie der Krieger,  
        der die Botschaft,  
            die den Sieg,  
                den die Griechen bei Marathon  
                    errungen hatten,  
                        verkündete,  
                            brachte,  
                                starb!
```

Pumping Lemma: Summary

- ▶ Every regular language is accepted by a DFA \mathcal{A} (with k states).
- ▶ Pumping lemma: words with at least k letters can be **pumped up**.
- ▶ If it is possible to pump up a word $w \in L$ and obtain a word $w' \notin L$, then L is **not regular**.
 - ▶ Make sure to handle quantifiers correctly!
- ▶ Practical relevance
 - ▶ FAs cannot **count arbitrarily high**.
 - ▶ **Nested structures** are not regular.
 - ▶ programming languages
 - ▶ natural languages
 - ▶ More powerful tools are needed to handle these languages.

Outline

Introduction

Regular Languages and Finite Automata

Regular Expressions

Finite Automata

The Pumping Lemma

Properties of Regular Languages

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

Regular languages: Closure properties

Reminder:

- ▶ **Formal languages** are sets of words (over a finite alphabet)
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 - ▶ There exists a DFA \mathcal{A} with $L(\mathcal{A}) = L$
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Question

What can we do to regular languages and be sure the result is still regular?

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Meta-Question: How do we answer these questions?

Closure properties (Theorem)

Theorem (Closure properties of regular languages)

Let L_1 and L_2 be regular languages. Then the following languages are also regular:

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Proof.

Idea: using (disjoint) finite automata for L_1 and L_2 , construct an automaton for the different languages above. □

Closure under union, concatenation, and Kleene-star

We use the same construction that was used to generate NFAs for regular expressions:

Let \mathcal{A}_{L_1} and \mathcal{A}_{L_2} be automata for L_1 and L_2 .

$L_1 \cup L_2$ new initial and final states,

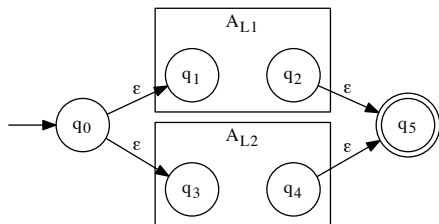
ε -transitions to initial/final states of \mathcal{A}_{L_1} and \mathcal{A}_{L_2}

$L_1 \cdot L_2$ ε -transition from final state of \mathcal{A}_{L_1} to initial state of \mathcal{A}_{L_2}

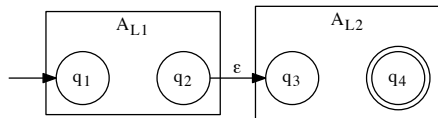
- $(L_1)^*$
- ▶ new initial and final states (with ε -transitions),
 - ▶ ε -transitions from the original final states to the original initial state,
 - ▶ ε -transition from the new initial to the new final state.

Visual refresher

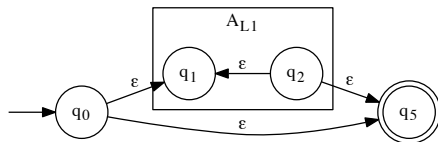
$L_1 \cup L_2$



$L_1 \circ L_2$



L_1^*



Closure under intersection

Let $\mathcal{A}_{L_1} = (Q_1, \Sigma, \delta_1, q_{0_1}, F_1)$ and $\mathcal{A}_{L_2} = (Q_2, \Sigma, \delta_2, q_{0_2}, F_2)$ be DFAs for L_1 and L_2 .

An automaton $L = (Q, \Sigma, \delta, q_0, F)$ for $\mathcal{A}_{L_1} \cap \mathcal{A}_{L_2}$ can be generated as follows:

- ▶ $Q = Q_1 \times Q_2$
- ▶ $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$ for all $q_1 \in Q_1, q_2 \in Q_2, a \in \Sigma$
- ▶ $q_0 = (q_{0_1}, q_{0_2})$
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- ▶ $q_0 = (q_{0_1}, q_{0_2})$
- ▶ $F = F_1 \times F_2$

This **product automaton**

- ▶ starts in state that corresponds to initial states of \mathcal{A}_{L_1} and \mathcal{A}_{L_2} ,
- ▶ simulates simultaneous processing in both automata
- ▶ accepts if both \mathcal{A}_{L_1} and \mathcal{A}_{L_2} accept.

Exercise: Product automaton

Generate automata for

- ▶ $L_1 = \{w \in \{0, 1\}^* \mid |w|_1 \text{ is divisible by } 2\}$
- ▶ $L_2 = \{w \in \{0, 1\}^* \mid |w|_1 \text{ is divisible by } 3\}$

Then generate an automaton for $L_1 \cap L_2$.

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Solution

End lecture 8

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Theorem (Closure under complement)

Let L be a regular language over Σ . Then $\bar{L} = \Sigma^ \setminus L$ is regular.*

Let $\mathcal{A}_L = (Q, \Sigma, q_0, \delta, F)$ be a DFA for the language L .

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All we have to do is exchange accepting and non-accepting states.

Closure properties: exercise

Show that $L = \{w \in \{a, b\}^* \mid |w|_a = |w|_b\}$ is not regular.

Hint: Use the following:

- ▶ $a^n b^n$ is not regular. (Pumping lemma)
- ▶ $a^* b^*$ is regular. (Regular expression)
- ▶ (one of) the closure properties shown before.

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Every finite language, i.e. every language containing only a finite number of words, is regular.

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Let $L = \{w_1, \dots, w_n\}$.

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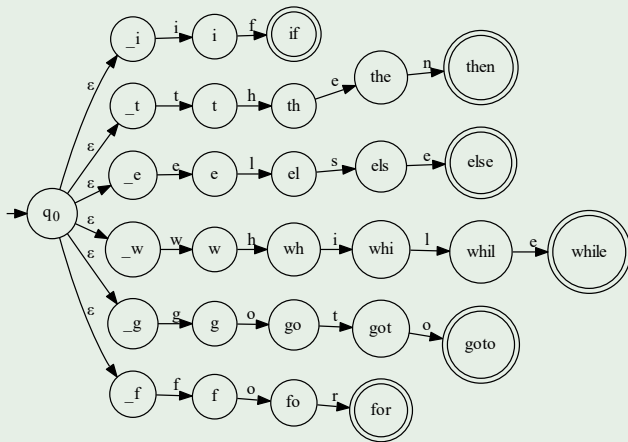
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Then the resulting automaton, with q_0 as initial state and all q_{f_i} as final states, accepts L . □

Example: finite language

Example ($L = \{if, then, else, while, goto, for\}$ over Σ_{ASCII})



Finite languages and regular expressions

Theorem (Regularity of finite languages)

Every finite language is regular.

Alternate proof.

Let $L = \{w_1, w_2, \dots, w_n\}$.

Write L as the regular expression $w_1 + w_2 + \dots + w_n$. □

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Corollary

The class of finite languages is characterised by

- ▶ *acyclic NFAs (or DFAs that have no cycles on any path from the initial state to an accepting state)*
- ▶ *regular expressions without Kleene star.*

Decision problems

For regular languages L_1 and L_2 and a word w , answer the following questions:

Is there a word in L_1 ? emptiness problem

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| Is w an element of L_1 ? | word problem |
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| Is L_1 finite? | finiteness problem |

Emptiness problem

Theorem (Emptiness problem for regular languages)

The emptiness problem for regular languages is decidable.

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Proof.

Algorithm: Let \mathcal{A} be an automaton accepting the language L .

- ▶ Starting with the initial state q_0 , mark all states to which there is a transition from q_0 as **reachable**.
- ▶ Continue with transitions from states which are already marked as **reachable** until either a final state is reached or no further states are reachable.
- ▶ If a final (accepting) state is **reachable**, then $L \neq \{\}$ holds.



Group exercise: Emptiness problem

- ▶ Find an alternative algorithm for the checking emptiness, using the results from the chapter on equivalence.

Word problem

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Proof.

Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ be a DFA accepting L and $w = c_1c_2 \dots c_n$.

Algorithm:

- ▶ $q_1 := \delta(q_0, c_1)$
- ▶ $q_2 := \delta(q_1, c_2)$
- ▶ ...
- ▶ If $q_n \in F$ holds, then \mathcal{A} accepts w .



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All we have to do is simulate the run of \mathcal{A} on w .

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Alternative proof.

One can also use closure properties and decidability of the emptiness problem:

$$L_1 = L_2 \text{ iff } \underbrace{(L_1 \cap \overline{L_2})}_{\text{words that are in } L_1, \text{ but not in } L_2} \cup \underbrace{(\overline{L_1} \cap L_2)}_{\text{words that are not in } L_1, \text{ but in } L_2} = \{\}$$



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Let \mathcal{A} be a DFA accepting L .

- ▶ Eliminate from \mathcal{A} all states that are not reachable from the initial state, obtaining \mathcal{A}_r .
- ▶ Eliminate from \mathcal{A}_r all states from which no final state is reachable, obtaining \mathcal{A}_f .
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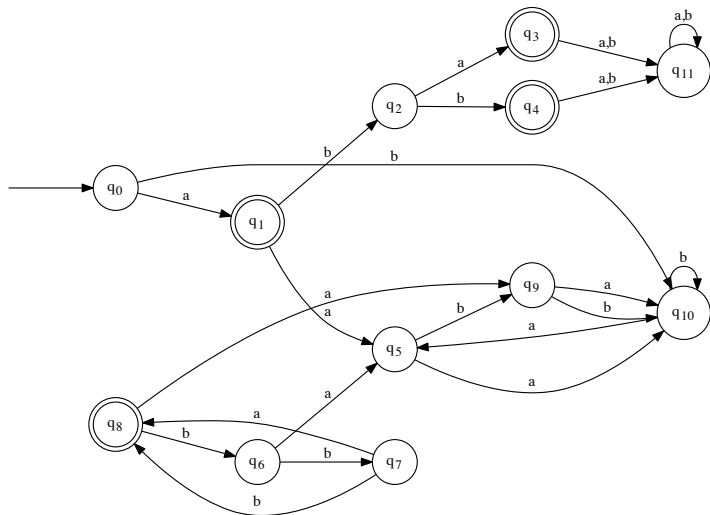
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Note that \mathcal{A}_f may be a NFA (because it misses some transitions)

Exercise: Finiteness

Consider the following DFA \mathcal{A} . Use the previous algorithm to decide if $L(\mathcal{A})$ is finite. Describe $L(\mathcal{A})$.



Regular languages: summary

Regular languages

- ▶ are characterised by
 - ▶ NFAs / DFAs
 - ▶ regular expressions
 - ▶ regular grammars
- ▶ can be transferred from one formalism to another one
- ▶ are **closed** under all operators (considered here)
- ▶ all decision problems (considered here) are **decidable**
- ▶ do not contain several interesting languages ($a^n b^n$, **counting**)
 - ▶ see chapter on **grammars**
- ▶ can express important features of programming languages
 - ▶ keywords
 - ▶ legal identifiers
 - ▶ numbers
- ▶ in compilers, these features are used by **scanners** (next chapter)

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Outline

Introduction

Regular Languages and Finite Automata

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

Computing Environment

- ▶ For practical exercises, you will need a complete Linux/UNIX environment. If you do not run one natively, there are several options:
 - ▶ You can install VirtualBox (<https://www.virtualbox.org>) and then install e.g. Ubuntu (<http://www.ubuntu.com/>) on a virtual machine. Make sure to install the *Guest Additions*
 - ▶ For Windows, you can install the **complete** UNIX emulation package Cygwin from <http://cygwin.com>
 - ▶ For MacOS, you can install `fink` (<http://fink.sourceforge.net/>) or MacPorts (<https://www.macports.org/>) and the necessary tools
- ▶ You will need at least `flex`, `bison`, `gcc`, `make`, and a good text editor

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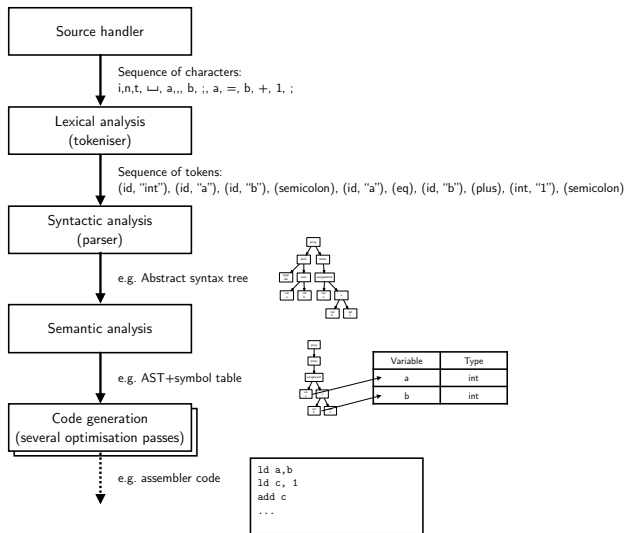
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High-Level Architecture of a Compiler



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- ▶ Handles input files
- ▶ Provides character-by-character access
- ▶ May maintain file/line/column (for error messages)
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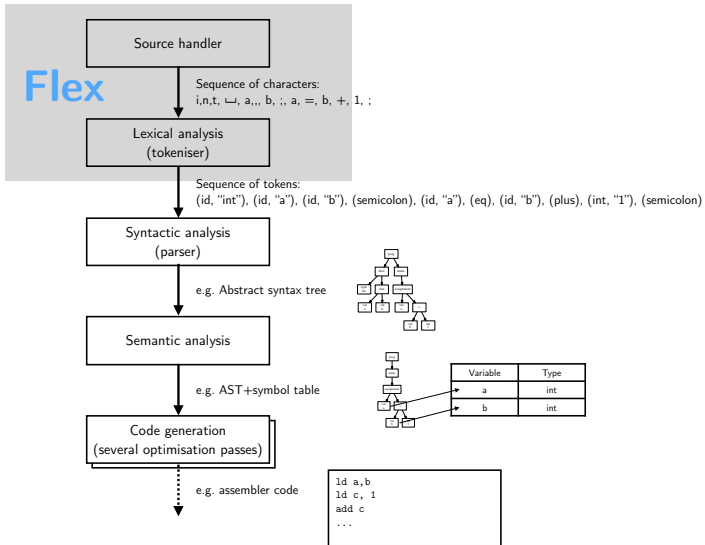
Result: Sequence of characters (with positions)

- ▶ Breaks program into **tokens**
- ▶ Typical tokens:
 - ▶ Reserved word (`if`, `while`)
 - ▶ Identifier (`i`, `database`)
 - ▶ Symbols (`{`, `}`, `(`, `)`, `+`, `-`, `...`)

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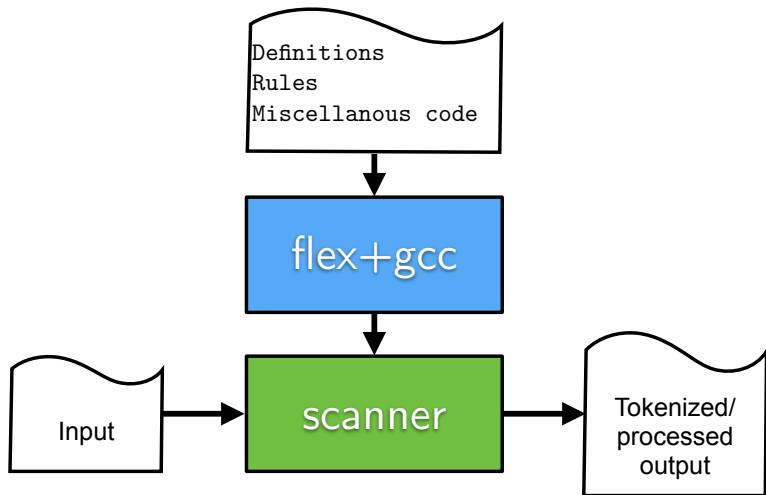
Automatisation with Flex



Flex Overview

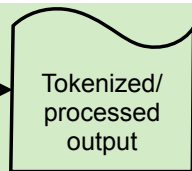
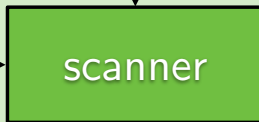
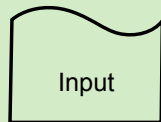
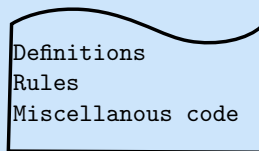
- ▶ Flex is a **scanner generator**
- ▶ Input: Specification of a regular language and what to do with it
 - ▶ Definitions - named regular expressions
 - ▶ Rules - patterns+actions
 - ▶ (miscellaneous support code)
- ▶ Output: Source code of **scanner**
 - ▶ Scans input for patterns
 - ▶ Executes associated actions
 - ▶ Default action: Copy input to output
 - ▶ Interface for higher-level processing: `yylex()` function

Flex Overview



Flex Overview

Development time



Execution time

Flex Example Task

- ▶ Goal: Sum up all numbers in a file, separately for ints and floats
- ▶ Given: A file with numbers and commands
 - ▶ Ints: Non-empty sequences of digits
 - ▶ Floats: Non-empty sequences of digits, followed by decimal dot, followed by (potentially empty) sequence of digits
 - ▶ Command `print`: Print current sums
 - ▶ Command `reset`: Reset sums to 0.
- ▶ At end of file, print sums

Flex Example Output

Input

```
12 3.1415
0.33333
print reset
2 11
1.5 2.5 print
1
print 1.0
```

Output

```
int: 12 ("12")
float: 3.141500 ("3.1415")
float: 0.333330 ("0.33333")
Current: 12 : 3.474830
Reset
int: 2 ("2")
int: 11 ("11")
float: 1.500000 ("1.5")
float: 2.500000 ("2.5")
Current: 13 : 4.000000
int: 1 ("1")
Current: 14 : 4.000000
float: 1.000000 ("1.0")
Final 14 : 5.000000
```

Basic Structure of Flex Files

- ▶ Flex files have 3 sections
 - ▶ Definitions
 - ▶ Rules
 - ▶ User Code
- ▶ Sections are separated by `%%`
- ▶ Flex files traditionally use the suffix `.l`

Example Code (definition section)

```
%option noyywrap  
  
DIGIT    [0-9]  
  
%{  
    int intval    = 0;  
    double floatval = 0.0;  
}%  
  
%%
```

Example Code (rule section)

```
{DIGIT}+    {
    printf( "int:   %d (\"%s\")\n", atoi(yytext), yytext );
    intval += atoi(yytext);
}
{DIGIT}+"."{DIGIT}*    {
    printf( "float: %f (\"%s\")\n", atof(yytext),yytext );
    floatval += atof(yytext);
}
reset {
    intval = 0;
    floatval = 0;
    printf("Reset\n");
}
print {
    printf("Current: %d : %f\n", intval, floatval);
}
\n|. {
    /* Skip */
}
```

Example Code (user code section)

```
%%  
int main( int argc, char **argv )  
{  
    ++argv, --argc; /* skip over program name */  
    if ( argc > 0 )  
        yyin = fopen( argv[0], "r" );  
    else  
        yyin = stdin;  
  
    yylex();  
  
    printf("Final   %d : %f\n", intval, floatval);  
}
```


Generating a scanner

```
> flex -t numbers.l > numbers.c
> gcc -c -o numbers.o numbers.c
> gcc numbers.o -o scan_numbers
> ./scan_numbers Numbers.txt
int: 12 ("12")
float: 3.141500 ("3.1415")
float: 0.333330 ("0.33333")
Current: 12 : 3.474830
Reset
int: 2 ("2")
int: 11 ("11")
float: 1.500000 ("1.5")
float: 2.500000 ("2.5")
...
```

Flexing in detail

```
> flex -tv numbers.l > numbers.c
scanner options: -tvI8 -Cem
37/2000 NFA states
18/1000 DFA states (50 words)
5 rules
Compressed tables always back-up
1/40 start conditions
20 epsilon states, 11 double epsilon states
6/100 character classes needed 31/500 words
of storage, 0 reused
36 state/nextstate pairs created
24/12 unique/duplicate transitions
...
381 total table entries needed
```

Exercise: Building a Scanner

- ▶ Download the `flex` example and input from <http://www.lehre.dhbw-stuttgart.de/~sschulz/fla2018.html>
- ▶ Build and execute the program:
 - ▶ Generate the scanner with `flex`
 - ▶ Compile/link the C code with `gcc`
 - ▶ Execute the resulting program in the input file

Definition Section

- ▶ Can contain `flex` options
- ▶ Can contain (C) initialization code
 - ▶ Typically `#include()` directives
 - ▶ Global variable definitions
 - ▶ Macros and type definitions
 - ▶ Initialization code is embedded in `%{` and `%}`
- ▶ Can contain definitions of regular expressions
 - ▶ Format: `NAME RE`
 - ▶ Defined `NAMES` can be referenced later

Regular Expressions in Practice (1)

- ▶ The minimal syntax of REs as discussed before suffices to show their equivalence to finite state machines
- ▶ Practical implementations of REs (e.g. in Flex) use a richer and more powerful syntax
- ▶ Regular expressions in Flex are based on the ASCII alphabet
- ▶ We distinguish between the set of operator symbols

$$O = \{., *, +, ?, -, \sim, |, (,), [,], \{, \}, <, >, /, \backslash, \wedge, \$, \}$$

and the set of regular expressions

1. $c \in \Sigma_{\text{ASCII}} \setminus O \longrightarrow c \in R$
2. $“.” \in R$
any character but newline ($\backslash n$)

Regular Expressions in Practice (2)

3. $x \in \{a, b, f, n, r, t, v\} \longrightarrow \backslash x \in R$
defines the following control characters
- $\backslash a$ (alert)
 - $\backslash b$ (backspace)
 - $\backslash f$ (form feed)
 - $\backslash n$ (newline)
 - $\backslash r$ (carriage return)
 - $\backslash t$ (tabulator)
 - $\backslash v$ (vertical tabulator)
4. $a, b, c \in \{0, \dots, 7\} \longrightarrow \backslash abc \in R$ octal representation of a character's ASCII code (e.g. $\backslash 040$ represents the empty space “ ”)

Regular Expressions in Practice (3)

5. $c \in O \longrightarrow \backslash c \in R$
escaping operator symbols
6. $r_1, r_2 \in R \longrightarrow r_1 r_2 \in R$
concatenation
7. $r_1, r_2 \in R \longrightarrow r_1 | r_2 \in R$
infix operation using “|” rather than “+”
8. $r \in R \longrightarrow r^* \in R$
Kleene star
9. $r \in R \longrightarrow r^+ \in R$
(one or more of r)
10. $r \in R \longrightarrow r? \in R$
optional presence (zero or one r)

Regular Expressions in Practice (4)

- $r \in R, n \in \mathbb{N} \rightarrow r\{n\} \in R$
concatenation of n times r
- $r \in R; m, n \in \mathbb{N}; m \leq n \rightarrow r\{m, n\} \in R$
concatenation of between m and n times r
- $r \in R \rightarrow \hat{r} \in R$
 r has to be at the **beginning** of line
- $r \in R \rightarrow r\$ \in R$
 r has to be at the **end** of line
- $r_1, r_2 \in R \rightarrow r_1/r_2 \in R$
The same as r_1r_2 , however, only the contents of r_1 is consumed.
The **trailing context** r_2 can be processed by the next rule.
- $r \in R \rightarrow (r) \in R$
Grouping regular expressions with brackets.

17. Ranges

- $[aeiou] \doteq a|e|i|o|u$
- $[a-z] \doteq a|b|c|\dots|z$
- $[a-zA-Z0-9]$: alphanumeric characters
- $[\^0-9]$: all ASCII characters w/o digits

18. $[] \in R$

empty space

19. $w \in \{\Sigma_{\text{ASCII}} \setminus \{\backslash, \text{"}\}\}^* \longrightarrow \text{"}w\text{"} \in R$
verbatim text (no escape sequences)

21. $r \in R \longrightarrow \sim r \in R$

The `upto` operator matches the **shortest** string ending with r .

22. predefined character classes

- ▶ `[:alnum:]` `[:alpha:]` `[:blank:]`
- ▶ `[:cntrl:]` `[:digit:]` `[:graph:]`
- ▶ `[:lower:]` `[:print:]` `[:punct:]`
- ▶ `[:space:]` `[:upper:]` `[:xdigit:]`

Regular Expressions in Practice (precedences)

- I. “(”, “)” (strongest)
- II. “*”, “+”, “?”
- III. concatenation
- IV. “|” (weakest)

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Example

$a*b | c+de \doteq ((a*)b) | (((c+)d)e)$

Regular Expressions in Practice (precedences)

- I. “(”, “)” (strongest)
- II. “*”, “+”, “?”
- III. concatenation
- IV. “|” (weakest)

Example

$a^*b | c+de \doteq ((a^*)b) | (((c+)d)e)$

**Rule of thumb: *, +, ? bind the smallest possible RE.
Use () if in doubt!**

Regular Expressions in Practice (definitions)

- ▶ Assume definition `NAME DEF`
 - ▶ In later REs. `{NAME}` is expanded to `(DEF)`
- ▶ Example:

```
DIGIT  [0-9]
INTEGER {DIGIT}+
PAIR    \({INTEGER}, {INTEGER}\)
```

Exercise: extended regular expressions

Given the alphabet Σ_{ascii} , how would you express the following practical REs using only the simple REs we have used so far?

1 [a-z]

2 [^0-9]

3 (r)+

4 (r){3}

5 (r){3,7}

6 (r)?

Example Code (definition section) (revisited)

```
%option noyywrap

DIGIT    [0-9]

%{
    int    intval    = 0;
    double floatval = 0.0;
}%

%%
```


Rule Section

- ▶ This is the core of the scanner!
- ▶ Rules have the form `PATTERN ACTION`
- ▶ Patterns are regular expressions
 - ▶ Typically use previous definitions
- ▶ There has to be white space between pattern and action
- ▶ Actions are C code
 - ▶ Can be embedded in `{` and `}`
 - ▶ Can be simple C statements
 - ▶ For a token-by-token scanner, must include `return` statement
 - ▶ Inside the action, the variable `yytext` contains the text matched by the pattern
 - ▶ Otherwise: Full input file is processed

Example Code (rule section) (revisited)

```
{DIGIT}+    {
    printf( "int:   %d (\"%s\")\n", atoi(yytext), yytext );
    intval += atoi(yytext);
}
{DIGIT}+"."{DIGIT}*    {
    printf( "float: %f (\"%s\")\n", atof(yytext),yytext );
    floatval += atof(yytext);
}
reset {
    intval = 0;
    floatval = 0;
    printf("Reset\n");
}
print {
    printf("Current: %d : %f\n", intval, floatval);
}
\n|. {
    /* Skip */
}
```

User code section

- ▶ Can contain all kinds of code
- ▶ For stand-alone scanner: must include `main()`
- ▶ In `main()`, the function `yylex()` will invoke the scanner
- ▶ `yylex()` will read data from the file pointer `yyin`
(so `main()` must set it up reasonably)

Example Code (user code section) (revisited)

```
%%  
int main( int argc, char **argv )  
{  
    ++argv, --argc; /* skip over program name */  
    if ( argc > 0 )  
        yyin = fopen( argv[0], "r" );  
    else  
        yyin = stdin;  
  
    yylex();  
  
    printf("Final   %d : %f\n", intval, floatval);  
}
```

A comment on comments

- ▶ Comments in Flex are complicated
 - ▶ ...because nearly everything can be a pattern
- ▶ Simple rules:
 - ▶ Use old-style C comments `/* This is a comment */`
 - ▶ Never start them in the first column
 - ▶ Comments are copied into the generated code
 - ▶ Read the manual if you want the dirty details

▶ Flex online:

- ▶ `https://github.com/westes/flex`
- ▶ **Manual:** `https://westes.github.io/flex/manual/`
- ▶ **REs:** `https://westes.github.io/flex/manual/Patterns.html`

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- ▶ `make` knows flex

- ▶ **Make will automatically generate** `file.o` from `file.l`
- ▶ **Be sure to set** `LEX=flex` to enable flex extensions
- ▶ **Makefile example:**

```
LEX=flex
all: scan_numbers
numbers.o: numbers.l

scan_numbers: numbers.o
    gcc numbers.o -o scan_numbers
```

Flexercise (1)

A security audit firm needs a tool that scans documents for the following:

- ▶ Email addresses

- ▶ Format: String over `[A-Za-z0-9_~]`, followed by `@`, followed by a domain name according to RFC-1034, <https://tools.ietf.org/html/rfc1034>, Section 3.5 (we only consider the case that the domain name is not empty)

- ▶ (simplified) Web addresses

- ▶ `http://` followed by an RFC-1034 domain name, optionally followed by `:<port>` (where `<port>` is a non-empty sequence of digits), optionally followed by one or several parts of the form `/<str>`, where `<str>` is a non-empty sequence over `[A-Za-z0-9_~]`

Flexercise (2)

▶ Bank account numbers

- ▶ Old-style bank account numbers start with an identifying string, optionally followed by ., optionally followed by :, optionally followed by spaces, followed by a non-empty sequence of up to 10 digits. Identifying strings are `Konto`, `Kto`, `KNr`, `Ktonr`, `Kontonummer`
- ▶ (German) IBANs are strings starting with `DE`, followed by exactly 20 digits. Human-readable IBANs have spaces after every 4 characters (the last group has only 2 characters)

▶ Examples:

- ▶ `Rosenda@gidwd-39.at.z8o3rw2.zhv`
- ▶ `http://jzl.j51g.m-x95.vi5/ojlg_i1/72zz_gt68f`
- ▶ `http://iefbottw99.v4gy.zslu9q.zrc2es01nr.dy:8004`
- ▶ `Ktonr. 241524`
- ▶ `DE26959558703965641174`
- ▶ `DE27 0192 8222 4741 4694 55`

Flexercise (3)

- ▶ Create a programm scanning for the data described above and printing the found items.
- ▶ Example data for Jan Hladik's lecture can be found in `http://www.lehre.dhbw-stuttgart.de/~hladik/FLA/skim-source.txt`
- ▶ Example input/output data for Stephan Schulz's lecture can be found under `http://www.lehre.dhbw-stuttgart.de/~sschulz/fla2018.html`

Flexercise (3)

- ▶ Create a programm scanning for the data described above and printing the found items.
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End lecture 10

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So far, we have seen

- ▶ regular expressions: compact description of regular languages
- ▶ finite automata: recognise words of a regular language

Formal Grammars: Motivation

So far, we have seen

- ▶ regular expressions: compact description of regular languages
- ▶ finite automata: recognise words of a regular language

Another, more powerful formalism: formal grammars

- ▶ generate words of a language
- ▶ contain a set of rules allowing to replace symbols with different symbols

Example (Formal grammars)

$S \rightarrow aA, \quad A \rightarrow bB, \quad B \rightarrow \varepsilon$

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$S \rightarrow aA, \quad A \rightarrow bB, \quad B \rightarrow \varepsilon$

generates ab (starting from S): $S \rightarrow aA \rightarrow abB \rightarrow ab$

$S \rightarrow \varepsilon, \quad S \rightarrow aSb$

generates $a^n b^n$

Definition (Grammar according to Chomsky)

A (formal) grammar is a quadruple

$$G = (N, \Sigma, P, S)$$

with

- 1 the set of non-terminal symbols N ,
- 2 the set of terminal symbols Σ ,
- 3 the set of production rules P of the form

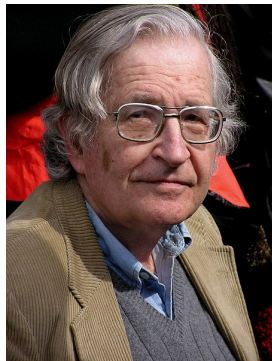
$$\alpha \rightarrow \beta$$

with $\alpha \in V^*NV^*$, $\beta \in V^*$, $V = N \cup \Sigma$

- 4 the distinguished start symbol $S \in N$.

Noam Chomsky (*1928)

- ▶ Linguist, philosopher, logician, . . .
- ▶ BA, MA, PhD (1955) at the University of Pennsylvania
- ▶ Mainly teaching at MIT (since 1955)
 - ▶ Also Harvard, Columbia University, Institute of Advanced Studies (Princeton), UC Berkely, . . .
- ▶ Opposition to Vietnam War, Essay *The Responsibility of Intellectuals*
- ▶ Most cited academic (1980-1992)
- ▶ “World’s top public intellectual” (2005)
- ▶ More than 40 honorary degrees



Example (C identifiers)

$G = (N, \Sigma, P, S)$ describes C identifiers:

- ▶ alpha-numeric words
- ▶ which must not start with a digit
- ▶ and may contain an underscore (`_`)

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$N = \{S, R, L, D\}$ (start, rest, letter, digit),

$\Sigma = \{a, \dots, z, A, \dots, Z, 0, \dots, 9, _ \}$,

$P = \{$

S	\rightarrow	$LR _R$
R	\rightarrow	$LR DR _R \varepsilon$
L	\rightarrow	$a \dots z A \dots Z$
D	\rightarrow	$0 \dots 9\}$

Grammar for C identifiers

Example (C identifiers)

$G = (N, \Sigma, P, S)$ describes C identifiers:

- ▶ alpha-numeric words
- ▶ which must not start with a digit
- ▶ and may contain an underscore (`_`)

$N = \{S, R, L, D\}$ (start, rest, letter, digit),

$\Sigma = \{a, \dots, z, A, \dots, Z, 0, \dots, 9, _ \}$,

$$P = \left\{ \begin{array}{l} S \rightarrow LR_R \\ R \rightarrow LR|DR|_R|\varepsilon \\ L \rightarrow a|\dots|z|A|\dots|Z \\ D \rightarrow 0|\dots|9 \end{array} \right.$$

$\alpha \rightarrow \beta_1 | \dots | \beta_n$ is an abbreviation for $\alpha \rightarrow \beta_1, \dots, \alpha \rightarrow \beta_n$.

Formal grammars: derivation, language

Definition (Derivation, Language of a Grammar)

For a grammar $G = (N, \Sigma, P, S)$ and words $x, y \in (\Sigma \cup N)^*$, we say that

G derives y from x in one step $(x \Rightarrow_G y)$ iff

$$\exists u, v, p, q \in V^* : (x = upv) \wedge (p \rightarrow q \in P) \wedge (y = uqv)$$

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Moreover, we say that

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with $w_0 = x, w_n = y, w_{i-1} \Rightarrow_G w_i$ for $i \in \{1, \dots, n\}$

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The language of G is $L(G) = \{w \in \Sigma^* \mid S \Rightarrow_G^* w\}$

Example (G_3)

Let $G_3 = (N, \Sigma, P, S)$ with

- ▶ $N = \{S\}$,
- ▶ $\Sigma = \{a\}$,
- ▶ $P = \{S \rightarrow aS, \quad S \rightarrow \varepsilon\}$.

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Derivations of G_3 have the general form

$$S \Rightarrow aS \Rightarrow aaS \Rightarrow \dots \Rightarrow a^n S \Rightarrow a^n$$

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Derivations of G_3 have the general form

$$S \Rightarrow aS \Rightarrow aaS \Rightarrow \dots \Rightarrow a^n S \Rightarrow a^n$$

The language produced by G_3 is

$$L(G_3) = \{a^n \mid n \in \mathbb{N}\}.$$

Example (G_2)

Let $G_2 = (N, \Sigma, P, S)$ with

- ▶ $N = \{S\}$,
- ▶ $\Sigma = \{a, b\}$,
- ▶ $P = \{S \rightarrow aSb, S \rightarrow \varepsilon\}$

Grammars and derivations (cont')

Example (G_2)

Let $G_2 = (N, \Sigma, P, S)$ with

- ▶ $N = \{S\}$,
- ▶ $\Sigma = \{a, b\}$,
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Derivations of G_2 :

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow \dots \Rightarrow a^n S b^n \Rightarrow a^n b^n.$$

Grammars and derivations (cont')

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Grammars and derivations (cont')

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Derivations of G_2 :

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow \dots \Rightarrow a^n S b^n \Rightarrow a^n b^n.$$

$$L(G_2) = \{a^n b^n \mid n \in \mathbb{N}\}.$$

Reminder: $L(G_2)$ is not regular!

Example (G_1)

Let $G_1 = (N, \Sigma, P, S)$ with

▶ $N = \{S, B, C\}$,

▶ $\Sigma = \{a, b, c\}$,

▶ P :

$$S \rightarrow aSBC \quad 1$$

$$S \rightarrow aBC \quad 2$$

$$CB \rightarrow BC \quad 3$$

$$aB \rightarrow ab \quad 4$$

$$bB \rightarrow bb \quad 5$$

$$bC \rightarrow bc \quad 6$$

$$cC \rightarrow cc \quad 7$$

Exercise: Derivations in G_1

Example (G_1)

Let $G_1 = (N, \Sigma, P, S)$ with

▶ $N = \{S, B, C\}$,

▶ $\Sigma = \{a, b, c\}$,

▶ P :

$$S \rightarrow aSBC \quad 1$$

$$S \rightarrow aBC \quad 2$$

$$CB \rightarrow BC \quad 3$$

$$aB \rightarrow ab \quad 4$$

$$bB \rightarrow bb \quad 5$$

$$bC \rightarrow bc \quad 6$$

$$cC \rightarrow cc \quad 7$$

- ▶ Give derivations for 3 different words in $L(G_1)$
- ▶ Can you characterize $L(G_1)$?

Grammars and derivations (cont.)

Derivations of G_1 :

$$\begin{aligned} S &\Rightarrow_1 aSBC \Rightarrow_1 aaSBCBC \Rightarrow_1 \cdots \Rightarrow_1 a^{n-1}S(BC)^{n-1} \Rightarrow_2 a^n(BC)^n \\ &\Rightarrow_3^* a^n B^n C^n \Rightarrow_{4,5}^* a^n b^n C^n \Rightarrow_{6,7}^* a^n b^n c^n \end{aligned}$$

Grammars and derivations (cont.)

Derivations of G_1 :

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$$L(G_1) = \{a^n b^n c^n \mid n \in \mathbb{N}; n > 0\}.$$

Grammars and derivations (cont.)

Derivations of G_1 :

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$$L(G_1) = \{a^n b^n c^n \mid n \in \mathbb{N}; n > 0\}.$$

- ▶ These three derivation examples represent different classes of grammars or languages characterized by different properties.
- ▶ A widely used classification scheme of formal grammars and languages is the [Chomsky hierarchy](#) (1956).

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The Chomsky hierarchy (0)

Definition (Grammar of type 0)

Every Chomsky grammar $G = (N, \Sigma, P, S)$ is of **Type 0** or **unrestricted**.

The Chomsky hierarchy (1)

Definition (context-sensitive grammar)

A Chomsky grammar $G = (N, \Sigma, P, S)$ is of is **Type 1** (**context-sensitive**) if all productions are of the form

$$\alpha \rightarrow \beta \quad \text{with} \quad |\alpha| \leq |\beta|$$

Exception: the rule $S \rightarrow \varepsilon$ is allowed if S does not appear on the right-hand side of any rule

The Chomsky hierarchy (1)

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$$\alpha \rightarrow \beta \quad \text{with} \quad |\alpha| \leq |\beta|$$

Exception: the rule $S \rightarrow \varepsilon$ is allowed if S does not appear on the right-hand side of any rule

- ▶ Rules never derive shorter words
 - ▶ except for the empty word in the first step

Context-sensitive vs. monotonic grammars

- ▶ Grammars of the type defined on the last slide were originally called **monotonic** or **non-contracting** by Chomsky
- ▶ **Context-sensitive** grammars additionally have to satisfy:

$$\alpha_1 A \alpha_2 \rightarrow \alpha_1 \beta \alpha_2 \text{ with } A \in N; \alpha_1, \alpha_2 \in V^*, \beta \in VV^*$$

- ▶ rule application can depend on a context α_1, α_2
- ▶ context cannot be modified (or moved)
- ▶ only **one NTS** can be modified
- ▶ every monotonic grammar can be rewritten as context-sensitive
 - ▶ $AB \rightarrow BA$ is not context-sensitive, but $AB \rightarrow AY \rightarrow XY \rightarrow XA \rightarrow BA$
 - ▶ if terminal symbols are involved: replace $S \rightarrow aB \rightarrow ba$ with $S \rightarrow N_a B \rightarrow \dots N_b N_a \rightarrow b N_a \rightarrow ba$
- ▶ since context is irrelevant for the language class, we drop the context requirement for this lecture
- ▶ since the term “context-sensitive” is generally used in the literature, we stick with this term (for both grammars and languages)

The Chomsky hierarchy (2)

Definition (context-free grammar)

A Chomsky grammar $G = (N, \Sigma, P, S)$ is of is **Type 2 (context-free)** if all productions are of the form

$$A \rightarrow \beta \text{ with } A \in N; \beta \in V^*$$

The Chomsky hierarchy (2)

Definition (context-free grammar)

A Chomsky grammar $G = (N, \Sigma, P, S)$ is of is **Type 2 (context-free)** if all productions are of the form

$$A \rightarrow \beta \text{ with } A \in N; \beta \in V^*$$

- ▶ Only single non-terminals are replaced
 - ▶ independent of their context
- ▶ Contracting rules are **allowed!**
 - ▶ context-free grammars are **not** a subset of context-sensitive grammars
 - ▶ but: context-free **languages** are a subset of context-sensitive **languages**
 - ▶ reason: contracting rules can be removed from context-free grammars, but not from context-sensitive ones

The Chomsky hierarchy (3)

Definition (right-linear grammar)

A Chomsky grammar $G = (N, \Sigma, P, S)$ is of **Type 3** (right-linear or regular) if all productions are of the form

$$A \rightarrow aB$$

with $A \in N$; $B \in N \cup \{\varepsilon\}$; $a \in \Sigma \cup \{\varepsilon\}$

The Chomsky hierarchy (3)

Definition (right-linear grammar)

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with $A \in N; B \in N \cup \{\varepsilon\}; a \in \Sigma \cup \{\varepsilon\}$

- ▶ only one NTS on the left
- ▶ on the right: one TS, one NTS, both, or neither
- ▶ analogy with automata is obvious

Definition (language classes)

A language is called

recursively enumerable, context-sensitive, context-free, or regular,

if it can be generated by a

unrestricted, context-sensitive, context-free, or regular

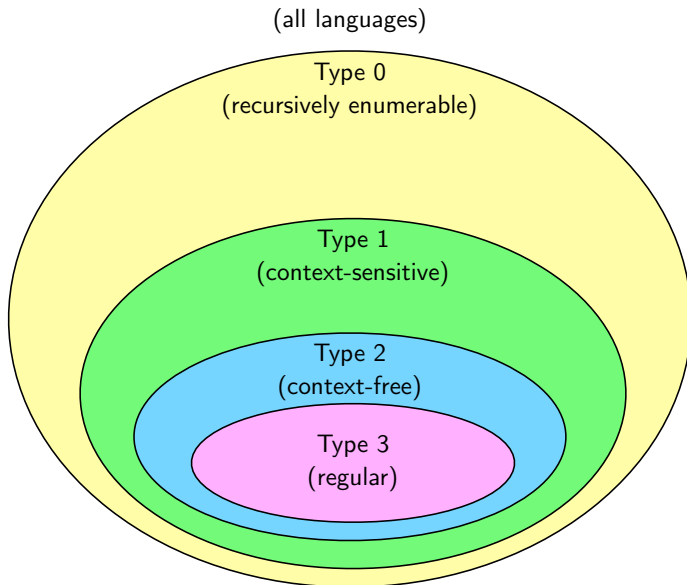
grammar, respectively.

Formal grammars vs. formal languages vs. machines

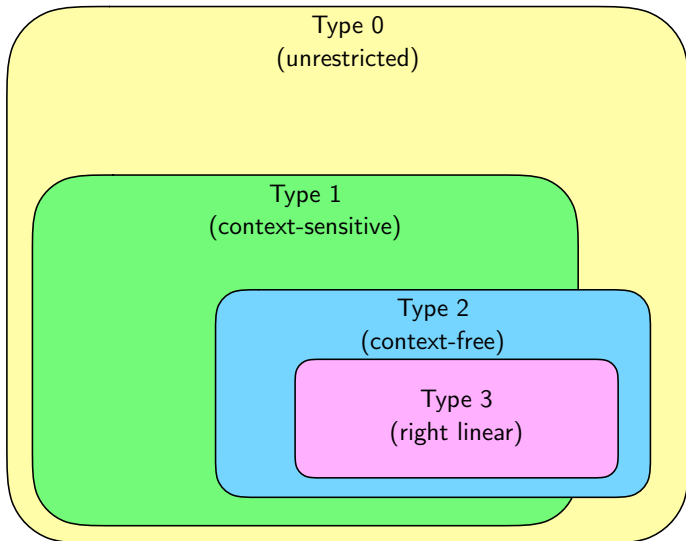
For each grammar/language type, there is a corresponding type of machine model:

grammar	language	machine
Type 0 unrestricted	recursively enumerable	Turing machine
Type 1	context-sensitive	linear-bounded non-deterministic Turing machine
Type 2	context-free	non-deterministic pushdown automaton
Type 3 right linear	regular	finite automaton

The Chomsky Hierarchy for Languages



The Chomsky Hierarchy for Grammars



The Chomsky hierarchy: examples

Example (C identifiers revisited)

$$S \rightarrow LR_R$$
$$R \rightarrow LR|DR_R|\varepsilon$$
$$L \rightarrow a|\dots|z|A|\dots|Z$$
$$D \rightarrow 0|\dots|9$$

The Chomsky hierarchy: examples

Example (C identifiers revisited)

$$S \rightarrow LR_R$$

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This grammar is context-free but not regular.

The Chomsky hierarchy: examples

Example (C identifiers revisited)

$$S \rightarrow LR_R$$

$$R \rightarrow LR|DR_R|\varepsilon$$

$$L \rightarrow a|\dots|z|A|\dots|Z$$

$$D \rightarrow 0|\dots|9$$

This grammar is context-free but not regular.

An equivalent regular grammar:

$$S \rightarrow AR|\dots|ZR|aR|\dots|zR|_R$$

$$R \rightarrow AR|\dots|ZR|aR|\dots|zR|0R|\dots|9R|_R|\varepsilon$$

The Chomsky hierarchy: examples revisited

Returning to the three derivation examples:

- ▶ G_3 with $P = \{S \rightarrow aS, S \rightarrow \varepsilon\}$
 - ▶ G_3 is regular.
 - ▶ So is the produced language $L_3 = \{a^n \mid n \in \mathbb{N}\}$.
- ▶ G_2 with $P = \{S \rightarrow aSb, S \rightarrow \varepsilon\}$
 - ▶ G_2 is context-free.
 - ▶ So is the produced language $L_2 = \{a^n b^n \mid n \in \mathbb{N}\}$.
- ▶ G_1 with $P = \{S \rightarrow aSBC, S \rightarrow aBC, CB \rightarrow BC, \dots\}$
 - ▶ G_1 is context-sensitive.
 - ▶ So is the produced language $L_1 = \{a^n b^n c^n \mid n \in \mathbb{N}; n > 0\}$.

The Chomsky hierarchy: exercises

1 Let $G = (N, \Sigma, P, S)$ with

▶ $N = \{S, A, B\}$,

▶ $\Sigma = \{a\}$,

▶ P :

$S \rightarrow \varepsilon$ 1

$S \rightarrow ABA$ 2

$AB \rightarrow aa$ 3

$aA \rightarrow aaaA$ 4

$A \rightarrow a$ 5

- What is G 's highest type?
- Show how G derives the word $aaaaa$.
- Formally describe the language $L(G)$.
- Define a regular grammar G' equivalent to G .

The Chomsky hierarchy: exercises (cont.)

- 2 An **octal constant** is a finite sequence of digits starting with 0 followed by at least one digit ranging from 0 to 7. Define a regular grammar encoding exactly the set of possible octal constants.

The Chomsky hierarchy: exercises (cont.)

3 Let $G = (N, \Sigma, P, S)$ with

▶ $N = \{S, A, B\},$

▶ $\Sigma = \{a, b, t\},$

▶ $P: S \rightarrow aAS \quad 1$

$S \rightarrow bBS \quad 2$

$S \rightarrow t \quad 3$

$At \rightarrow ta \quad 4$

$Bt \rightarrow tb \quad 5$

$Aa \rightarrow aA \quad 6$

$Ab \rightarrow bA \quad 7$

$Ba \rightarrow aB \quad 8$

$Bb \rightarrow bB \quad 9$

- a) What is G 's highest type?
b) Formally describe the language $L(G)$.

The Chomsky hierarchy: exercises (cont.)

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End lecture 11

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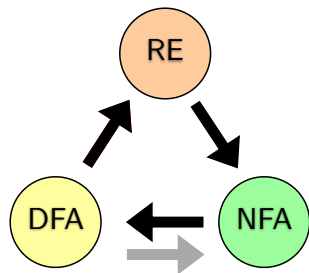
Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

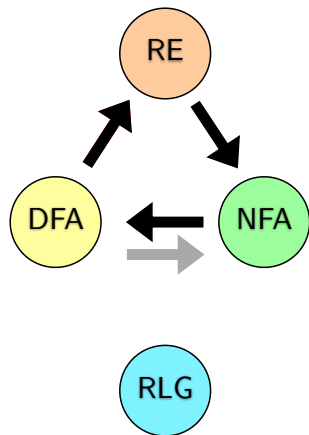
Right-linear grammars and regular languages



► We know:

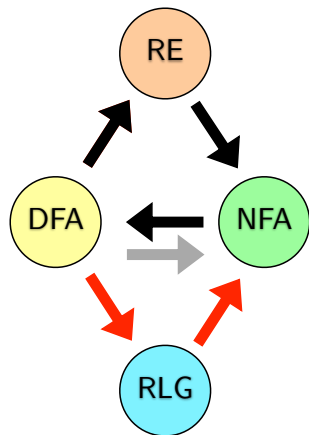
- Regular expression can be converted to NFAs
- NFAs can be converted to DFAs
- DFAs can be described by regular expressions
- Hence NFA, DFA, RE are equivalent and all describe the class of **regular languages**

Right-linear grammars and regular languages



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Right-linear grammars and regular languages



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 - ▶ Hence NFA, DFA, RE are equivalent and all describe the class of **regular languages**
- ▶ Now: Right-linear grammars
- ▶ We show:
 - ▶ DFAs can be converted into right-linear grammars
 - ▶ Right-linear grammars can be converted into NFAs

Regular languages and right-linear grammars

Theorem (right-linear grammars and regular languages)

The class of regular languages (generated by regular expressions, accepted by finite automata) is exactly the class of languages generated by right-linear grammars.

Regular languages and right-linear grammars

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The class of regular languages (generated by regular expressions, accepted by finite automata) is exactly the class of languages generated by right-linear grammars.

Proof.

We constructively prove the theorem by providing algorithms to...

- ▶ Convert a DFA to a right-linear grammar
- ▶ Convert right-linear grammar to NFA

... such that the languages of grammar and automaton are the same. □

DFA \rightsquigarrow right-linear grammar

Algorithm for transforming a DFA

$$\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$$

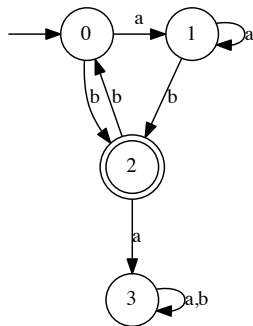
into a grammar

$$G = (N, \Sigma, P, S)$$

- ▶ $N = Q$
- ▶ $S = q_0$
- ▶ $P = \{p \rightarrow aq \mid ((p, a), q) \in \delta\} \cup \{p \rightarrow \varepsilon \mid p \in F\}$

Exercise: DFA to right-linear grammar

Consider the following DFA \mathcal{A} :



- Give a formal definition of \mathcal{A}
- Generate a right-linear grammar G with $L(G) = L(\mathcal{A})$

Right-linear grammar \rightsquigarrow NFA

Algorithm for transforming a grammar

$$G = (N, \Sigma, P, S)$$

into an NFA

$$\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$$

- ▶ $Q = N \cup \{q_f\}$ ($q_f \notin N$)
- ▶ $q_0 = S$
- ▶ $F = \{q_f\}$
- ▶ $\Delta = \{(A, c, B) \mid A \rightarrow cB \in P\} \cup$
 $\{(A, c, q_f) \mid A \rightarrow c \in P\} \cup$
 $\{(A, \varepsilon, B) \mid A \rightarrow B \in P\} \cup$
 $\{(A, \varepsilon, q_f) \mid A \rightarrow \varepsilon \in P\}$

Exercise: right-linear grammar to NFA

Transform the grammar $G = (\{S, A, B\}, \{a, b\}, P, S)$ into an NFA.

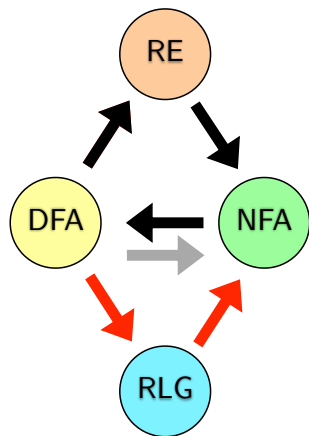
$$\begin{aligned}P : \quad S &\rightarrow aB \mid \varepsilon \\A &\rightarrow aB \mid b \\B &\rightarrow A\end{aligned}$$

Which language is generated by G ?

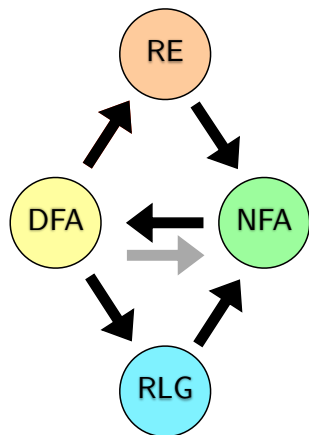
Right-linear grammars and regular languages

► We now know:

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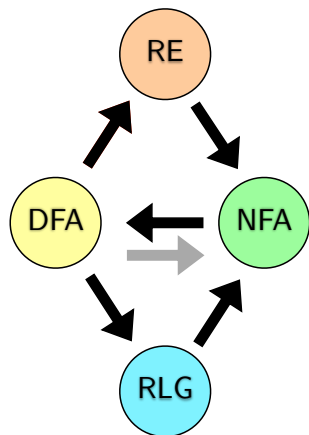
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- DFAs can be converted into right-linear grammars
- Right-linear grammars can be converted into NFAs

Right-linear grammars and regular languages



► We now know:

- Regular expression can be converted to NFAs
- NFAs can be converted to DFAs
- DFAs can be described by regular expressions
- Hence NFA, DFA, RE are equivalent and all describe the class of **regular languages**
- DFAs can be converted into right-linear grammars
- Right-linear grammars can be converted into NFAs

REs, DFAs, NFAs, Right-linear grammars are all equivalent formalisms do describe the class of regular languages!

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Context-free grammars

- ▶ Reminder: $G = (N, \Sigma, P, S)$ is context-free if all rules in P are of the form $A \rightarrow \beta$ with
 - ▶ $A \in N$ and
 - ▶ $\beta \in (\Sigma \cup N)^*$
- ▶ Context-free languages/grammars are highly relevant for practical applications
 - ▶ Core of most programming languages
 - ▶ XML
 - ▶ Algebraic expressions
 - ▶ Many aspects of human language

Grammars: equivalence and normal forms

Definition (equivalence)

Two grammars are called **equivalent** if they generate the same language.

Grammars: equivalence and normal forms

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We will now compute grammars that are equivalent to some given context-free grammar G but have “nicer” properties

- ▶ **Reduced** grammars contain no unproductive symbols
- ▶ Grammars in **Chomsky normal form** support efficient decision of the **word problem**

Grammars: equivalence and normal forms

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- ▶ **Reduced** grammars contain no unproductive symbols
- ▶ Grammars in **Chomsky normal form** support efficient decision of the **word problem**

I.e. grammars in CNF allow efficient parsing of arbitrary context-free languages!

Definition (reduced)

Let $G = (N, \Sigma, P, S)$ be a context-free grammar.

- ▶ $A \in N$ is called **terminating** if $A \Rightarrow_G^* w$ for some $w \in \Sigma^*$.
- ▶ $A \in N$ is called **reachable** if $S \Rightarrow_G^* uAv$ for some $u, v \in V^*$.
- ▶ G is called **reduced** if N contains only reachable and terminating symbols.

Terminating and reachable symbols

The terminating symbols can be computed as follows:

- 1 $T := \{A \in N \mid \exists w \in \Sigma^* : A \rightarrow w \in P\}$
- 2 add all symbols M to T with a rule $M \rightarrow D$ with $D \in (\Sigma \cup T)^*$
- 3 repeat step 2 until no further symbols can be added

Now T contains exactly the terminating symbols.

Terminating and reachable symbols

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Now T contains exactly the terminating symbols.

The reachable symbols can be computed as follows:

- 1 $R := \{S\}$
- 2 for every $A \in R$, add all symbols M with a rule $A \rightarrow V^*MV^*$
- 3 repeat step 2 until no further symbols can be added

Now R contains exactly the reachable symbols.

Reducing context-free grammars

Theorem (reduction of context-free grammars)

Every context-free grammar G can be transformed into an equivalent reduced context-free grammar G_r .

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Proof.

- 1 generate the grammar G_T by removing all **non-terminating** symbols (and rules containing them) from G
- 2 generate the grammar G_r by removing all **unreachable symbols** (and rules containing them) from G_T



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Sequence is important: symbols become unreachable if they only appear together with non-terminating symbols.

Example

Let $G = (N, \Sigma, P, S)$ with

▶ $N = \{S, A, B, C, T\},$

▶ $\Sigma = \{a, b, c\},$

▶ $P :$

$$S \rightarrow T|B|C$$

$$T \rightarrow AB$$

$$A \rightarrow a$$

$$B \rightarrow bB$$

$$C \rightarrow c$$

Reachable and terminating symbols

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Let $G = (N, \Sigma, P, S)$ with

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$$S \rightarrow T|B|C$$
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▶ terminating symbols in G : $C, A, S \rightsquigarrow G_T$

▶ reachable symbols in G_T : $S, C \rightsquigarrow G_r$

▶ note: A is still reachable in G !

Exercise: reducing grammars

Compute the reduced grammar $G = (N, \Sigma, P, S)$ for the following grammar $G' = (N', \Sigma, P', S)$:

1 $N' = \{S, A, B, C, D\},$

2 $\Sigma = \{a, b\},$

3 $P' :$

$$S \rightarrow A|aS|B$$

$$A \rightarrow a$$

$$A \rightarrow AS$$

$$A \rightarrow Ba$$

$$B \rightarrow Ba$$

$$C \rightarrow Da$$

$$D \rightarrow Cb$$

$$D \rightarrow a$$

Chomsky normal form

Reduced grammars can be further modified to allow for an efficient decision procedure for the word problem.

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Definition (CNF)

A context-free grammar (N, Σ, P, S) is in **Chomsky normal form** if all rules are of the kind

- ▶ $N \rightarrow a$ with $a \in \Sigma$
- ▶ $N \rightarrow AB$ with $A, B \in N$
- ▶ $S \rightarrow \epsilon$, if S does not appear on the right-hand side of any rule

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Transformation of a reduced grammar into CNF:

- 1 remove ε -productions
- 2 remove chain rules ($A \rightarrow B$)
- 3 introduce auxiliary symbols

Removal of ε -productions

Theorem (ε -free grammar)

Every context-free grammar can be transformed into an equivalent cf. grammar that does not contain rules of the kind $A \rightarrow \varepsilon$ (except $S \rightarrow \varepsilon$ if S does not appear on the rhs).

Removal of ε -productions

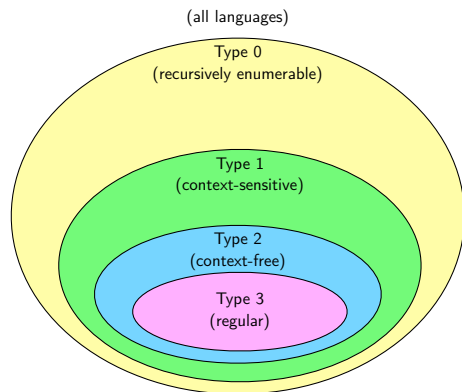
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Procedure:

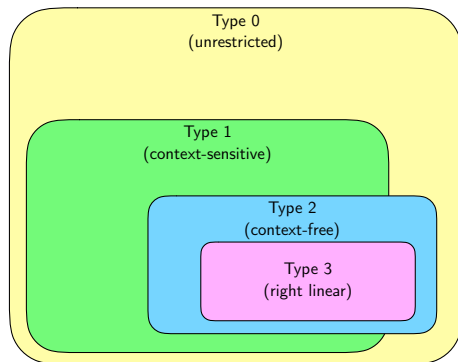
- 1 let $E = \{A \in N \mid A \rightarrow \varepsilon \in P\}$
- 2 add all symbols B to E for which there is a rule $B \rightarrow \beta$ with $\beta \in E^*$
- 3 repeat step 2 until no further symbols can be added
- 4 for every rule $C \rightarrow \beta_1 B \beta_2$ with $B \in E$
 - ▶ add rule $C \rightarrow \beta_1 \beta_2$ to P
 - ▶ repeat this process until no new rules are added
- 5 remove all rules $A \rightarrow \varepsilon$ from P
- 6 if $S \in E$
 - ▶ use a new start symbol S_0
 - ▶ add rules $S_0 \rightarrow \varepsilon | S$

Interlude: Chomsky-Hierarchy for Grammars (again)



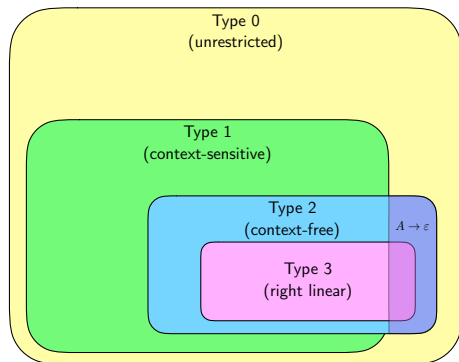
- ▶ For languages, Type-0, Type-1, Type-2, Type-3 form a real inclusion hierarchy

Interlude: Chomsky-Hierarchy for Grammars (again)



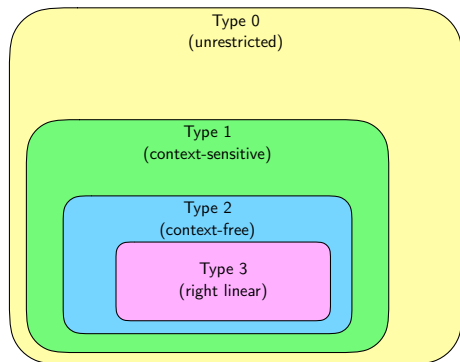
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- ▶ Not quite true for grammars:
 - ▶ $A \rightarrow \varepsilon$ allowed in context-free/regular grammars, not in context-sensitive grammars
- ▶ Eliminating ε -productions removes this discrepancy!

Removal of chain rules

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Procedure:

- 1 for every $A \in N$, compute the set $N(A) = \{B \in N \mid A \Rightarrow_G^* B\}$
(this can be done iteratively, as shown previously)
- 2 remove $A \rightarrow C$ for any $C \in N$ from P
- 3 add the following production rules to P
 $\{A \rightarrow w \mid w \notin N \text{ and } B \rightarrow w \in P \text{ and } B \in N(A)\}$

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Example

$A \rightarrow a|B; \quad B \rightarrow bb|C; \quad C \rightarrow ccc$

is equivalent to

$A \rightarrow a|bb|ccc; \quad B \rightarrow bb|ccc; \quad C \rightarrow ccc$

Chomsky normal form

Reminder: Chomsky normal form

A context-free grammar (N, Σ, P, S) is in CNF if all rules are of the kind

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Theorem (transformation into Chomsky normal form)

Every context free grammar can be transformed into an equivalent cf. grammar in Chomsky normal form.

Algorithm for computing Chomsky normal form

- 1 remove ε rules
- 2 remove chain rules
- 3 compute reduced grammar
 - 1 remove non-terminating symbols
 - 2 remove unreachable symbols
- 4 for all rules $A \rightarrow w$ with $w \notin \Sigma$:
 - ▶ for all $a \in \Sigma$ replace all occurrences of a in w by a new non-terminal symbol X_a
 - ▶ add rules $X_a \rightarrow a$
- 5 replace rules of the form $A \rightarrow B_1 B_2 \dots B_n$ with $n > 2$ with rules

$$\begin{aligned}A &\rightarrow B_1 C_1 \\C_1 &\rightarrow B_2 C_2 \\&\vdots \\C_{n-2} &\rightarrow B_{n-1} B_n\end{aligned}$$

where the C_i are new non-terminals

Exercise: transformation into CNF

Compute the Chomsky normal form of the following grammar:

$$G = (N, \Sigma, P, S)$$

▶ $N = \{S, A, B, C, D, E\}$

▶ $\Sigma = \{a, b\}$

▶ P :

$$S \rightarrow AB|SB|BDE$$

$$A \rightarrow Aa$$

$$B \rightarrow bB|BaB|ab$$

$$C \rightarrow SB$$

$$D \rightarrow E$$

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Solution

Chomsky NF: purpose

Why transform G into Chomsky NF?

- ▶ in a context-free grammar, derivations can have arbitrary length
 - ▶ if there are contracting rules, a derivation of w can contain words longer than w
 - ▶ if there are chain rules ($C \rightarrow B; B \rightarrow C$), a derivation of w can contain arbitrarily many steps
- ▶ **word problem** is difficult to decide
- ▶ if G is in CNF, for a word of length n , a derivation has $2n - 1$ steps:
 - ▶ $n - 1$ rule applications $A \rightarrow BC$
 - ▶ n rule applications $A \rightarrow a$
- ▶ word problem can be decided by checking all derivations of length $2n - 1$

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 - ▶ if there are chain rules ($C \rightarrow B; B \rightarrow C$), a derivation of w can contain arbitrarily many steps
- ▶ **word problem** is difficult to decide
- ▶ if G is in CNF, for a word of length n , a derivation has $2n - 1$ steps:
 - ▶ $n - 1$ rule applications $A \rightarrow BC$
 - ▶ n rule applications $A \rightarrow a$
- ▶ word problem can be decided by checking all derivations of length $2n - 1$
- ▶ That's still plenty of derivations!

Chomsky NF: purpose

Why transform G into Chomsky NF?

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More efficient algorithm: Cocke-Younger-Kasami (CYK)

Cocke-Younger-Kasami algorithm

- ▶ Efficient algorithm to decide the word problem for context-free grammars
- ▶ Core ideas independently developed by
 - ▶ John Cocke (1925—2002): *Programming languages and their compilers: Preliminary notes*, 1970 (with Jacob T. Schwartz)
 - ▶ Daniel H. Younger (??–): *Recognition and parsing of context-free languages in time n^3* , 1967
 - ▶ Tadao Kasami (1930–2007) *An efficient recognition and syntax-analysis algorithm for context-free languages*, 1965
- ▶ Complexity: $O(|w|^3 \cdot |G|)$
- ▶ Can provide *all* ways to parse/generate a word
- ▶ Extends to probabilistic parsing

CYK algorithm: idea

Decide the word problem for a context-free grammar G in Chomsky NF and a word w .

- ▶ find out which NTS are needed in the end to produce the TS for w (using production rules $A \rightarrow a$).
- ▶ iteratively find all NTS that can generate the required sequence of NTS (using production rules $A \rightarrow BC$).
- ▶ if S can produce the required sequence, $w \in L(G)$ holds.

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Mechanism:

- ▶ operates on a table.
- ▶ field in row i and column j contains all NTS that can generate the target word from character i through j .

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Mechanism:

- ▶ operates on a table.
- ▶ field in row i and column j contains all NTS that can generate the target word from character i through j .

Example of dynamic programming!

CYK algorithm: example

$S \rightarrow a$

$B \rightarrow b$

$B \rightarrow c$

$S \rightarrow SA$

$A \rightarrow BS$

$B \rightarrow BB$

$B \rightarrow BS$

$i \setminus j$	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						
$w =$	a	b	a	c	b	a

$w = abacba$

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CYK: formal algorithm

for $i := 1$ to n **do**

$N_{ii} := \{A \mid A \rightarrow a_i \in P\}$

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$j := i + d$

$N_{ij} := \{\}$

CYK: formal algorithm

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for $d := 1$ to $n - 1$ **do**

for $i := 1$ to $n - d$ **do**

$j := i + d$

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for $k := i$ to $j - 1$ **do**

$N_{ij} := N_{ij} \cup \{A \mid A \rightarrow BC \in P; B \in N_{ik}; C \in N_{(k+1)j}\}$

CYK algorithm: exercise

Consider the grammar
 $G = (N, \Sigma, P, S)$ from the previous
exercise

- ▶ $N = \{S, A, B, C\}$
- ▶ $\Sigma = \{a, b\}$

$$\begin{aligned}P : \quad S &\rightarrow AB|SB|BDE \\ A &\rightarrow Aa \\ B &\rightarrow bB|BaB|ab \\ C &\rightarrow SB \\ D &\rightarrow E \\ E &\rightarrow \varepsilon\end{aligned}$$

Use the CYK algorithm to determine if the following words can be generated by G :

- a) $w_1 = babaab$
- b) $w_2 = abba$

CYK algorithm: exercise

Consider the grammar
 $G = (N, \Sigma, P, S)$ from the previous
exercise

- ▶ $N = \{S, A, B, C_1, X_a, X_b\}$
- ▶ $\Sigma = \{a, b\}$

$$\begin{aligned}P : \quad S &\rightarrow SB|BC_1|X_bB|X_aX_b \\ B &\rightarrow BC_1|X_bB|X_aX_b \\ C_1 &\rightarrow X_aB \\ X_a &\rightarrow a \\ X_b &\rightarrow b\end{aligned}$$

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CYK algorithm: exercise

Consider the grammar
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- ▶ $N = \{S, A, B, D, X, Y\}$
- ▶ $\Sigma = \{a, b\}$

$$\begin{aligned}P : \quad S &\rightarrow SB|BD|YB|XY \\ B &\rightarrow BD|YB|XY \\ D &\rightarrow XB \\ X &\rightarrow a \\ Y &\rightarrow b\end{aligned}$$

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Outline

Introduction

Regular Languages and Finite Automata

Scanners and Flex

Formal Grammars and Context-Free Languages

Formal Grammars

The Chomsky Hierarchy

Right-linear Grammars

Context-free Grammars

Push-Down Automata

Properties of Context-free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

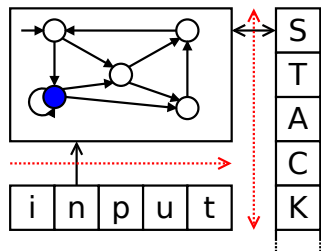
Bonus Exercises

Selected Solutions

Pushdown automata: motivation

- ▶ DFAs/NFAs are weaker than context-free grammars
- ▶ to accept languages like $a^n b^n$, an **unlimited storage component** is needed
- ▶ **Pushdown automata** have an unlimited **stack**
 - ▶ LIFO: last in, first out
 - ▶ only top symbol can be read
 - ▶ arbitrary amount of symbols can be added to the top

PDA: conceptual model



- ▶ extends FA by **unlimited stack**:
 - ▶ transitions can read and **write** stack
 - ▶ only a the top
 - ▶ **stack alphabet** Γ
 - ▶ **LIFO**: last in, first out
- ▶ acceptance condition
 - ▶ **empty stack** after reading input
 - ▶ no final states needed
- ▶ commonalities with FA:
 - ▶ read input from left to right
 - ▶ set of states, input alphabet
 - ▶ initial state

PDA transitions

$$\Delta \subseteq Q \times \Sigma \cup \{\epsilon\} \times \Gamma \times \Gamma^* \times Q$$

- ▶ PDA is in a state
- ▶ can read next input character or nothing
- ▶ must read (and remove) top stack symbol
- ▶ can write arbitrary amount of symbols on top of stack
- ▶ goes into a new state

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A transition (p, c, A, BC, q) can be written as follows:

$$p \quad c \quad A \quad \rightarrow \quad BC \quad q$$

Pushdown automata: definition

Definition (pushdown automaton)

A **pushdown automaton** (PDA) is a 6-tuple $(Q, \Sigma, \Gamma, \Delta, q_0, Z_0)$ where

- ▶ Q, Σ, q_0 are defined as for NFAs.
- ▶ Γ is the stack alphabet
- ▶ Z_0 is the initial stack symbol
- ▶ $\Delta \subseteq Q \times \Sigma \cup \{\varepsilon\} \times \Gamma \times \Gamma^* \times Q$ is the transition relation

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A **configuration** of a PDA is a triple (q, w, γ) where

- ▶ q is the current state
- ▶ w is the input yet unread
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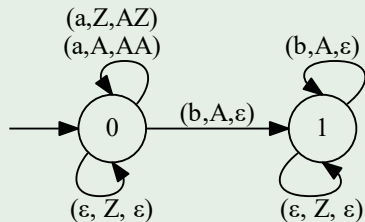
A PDA \mathcal{A} **accepts** a word $w \in \Sigma^*$ if, starting from the configuration (q_0, w, Z_0) , \mathcal{A} can reach the configuration $(q, \varepsilon, \varepsilon)$ for some q .

Example: PDA for $a^n b^n$

Example (Automaton \mathcal{A})

$\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, 0, Z)$

- ▶ $Q = \{0, 1\}$
- ▶ $\Sigma = \{a, b\}$
- ▶ $\Gamma = \{A, Z\}$
- ▶ $\Delta :$



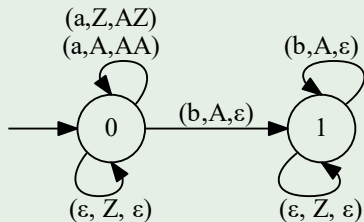
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Example (Automaton \mathcal{A})

$$\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, 0, Z)$$

- ▶ $Q = \{0, 1\}$
- ▶ $\Sigma = \{a, b\}$
- ▶ $\Gamma = \{A, Z\}$
- ▶ $\Delta :$

0	ε	Z	\rightarrow	ε	0	accept empty word
0	a	Z	\rightarrow	AZ	0	read first a, store A
0	a	A	\rightarrow	AA	0	read further a, store A
0	b	A	\rightarrow	ε	1	read first b, delete A
1	b	A	\rightarrow	ε	1	read further b, delete A
1	ε	Z	\rightarrow	ε	1	accept if all As have been deleted



PDA: example (2)

Process $aabb$:

0 ε Z \rightarrow ε 0

0 a Z \rightarrow AZ 0

0 a A \rightarrow AA 0

0 b A \rightarrow ε 1

1 b A \rightarrow ε 1

1 ε Z \rightarrow ε 1

PDA: example (2)

Process $aabb$:

1 $(0, aabb, Z)$

$0 \quad \varepsilon \quad Z \rightarrow \varepsilon \quad 0$

$0 \quad a \quad Z \rightarrow AZ \quad 0$

$0 \quad a \quad A \rightarrow AA \quad 0$

$0 \quad b \quad A \rightarrow \varepsilon \quad 1$

$1 \quad b \quad A \rightarrow \varepsilon \quad 1$

$1 \quad \varepsilon \quad Z \rightarrow \varepsilon \quad 1$

PDA: example (2)

Process $aabb$:

1 $(0, aabb, Z)$

2 $(0, abb, AZ)$

$0 \quad \varepsilon \quad Z \rightarrow \varepsilon \quad 0$

$0 \quad a \quad Z \rightarrow AZ \quad 0$

$0 \quad a \quad A \rightarrow AA \quad 0$

$0 \quad b \quad A \rightarrow \varepsilon \quad 1$

$1 \quad b \quad A \rightarrow \varepsilon \quad 1$

$1 \quad \varepsilon \quad Z \rightarrow \varepsilon \quad 1$

PDA: example (2)

Process $aabb$:

1 $(0, aabb, Z)$

2 $(0, abb, AZ)$

3 $(0, bb, AAZ)$

$0 \quad \varepsilon \quad Z \rightarrow \varepsilon \quad 0$

$0 \quad a \quad Z \rightarrow AZ \quad 0$

$0 \quad a \quad A \rightarrow AA \quad 0$

$0 \quad b \quad A \rightarrow \varepsilon \quad 1$

$1 \quad b \quad A \rightarrow \varepsilon \quad 1$

$1 \quad \varepsilon \quad Z \rightarrow \varepsilon \quad 1$

PDA: example (2)

0	ϵ	Z	\rightarrow	ϵ	0
0	a	Z	\rightarrow	AZ	0
0	a	A	\rightarrow	AA	0
0	b	A	\rightarrow	ϵ	1
1	b	A	\rightarrow	ϵ	1
1	ϵ	Z	\rightarrow	ϵ	1

Process $aabb$:

1 (0, $aabb$, Z)

2 (0, abb , AZ)

3 (0, bb , AAZ)

4 (1, b , AZ)

PDA: example (2)

0	ϵ	Z	\rightarrow	ϵ	0
0	a	Z	\rightarrow	AZ	0
0	a	A	\rightarrow	AA	0
0	b	A	\rightarrow	ϵ	1
1	b	A	\rightarrow	ϵ	1
1	ϵ	Z	\rightarrow	ϵ	1

Process $aabb$:

1 (0, $aabb$, Z)

2 (0, abb , AZ)

3 (0, bb , AAZ)

4 (1, b , AZ)

5 (1, ϵ , Z)

PDA: example (2)

0	ϵ	Z	\rightarrow	ϵ	0
0	a	Z	\rightarrow	AZ	0
0	a	A	\rightarrow	AA	0
0	b	A	\rightarrow	ϵ	1
1	b	A	\rightarrow	ϵ	1
1	ϵ	Z	\rightarrow	ϵ	1

Process $aabb$:

1 $(0, aabb, Z)$

2 $(0, abb, AZ)$

3 $(0, bb, AAZ)$

4 $(1, b, AZ)$

5 $(1, \epsilon, Z)$

6 $(1, \epsilon, \epsilon)$

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0	ε	Z	\rightarrow	ε	0
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0	b	A	\rightarrow	ε	1
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1	ε	Z	\rightarrow	ε	1

Process *aabb*:

1 (0, *aabb*, Z)

2 (0, *abb*, AZ)

3 (0, *bb*, AAZ)

4 (1, *b*, AZ)

5 (1, ε , Z)

6 (1, ε , ε)

Process *abb*:

1 (0, *abb*, Z)

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PDA: example (2)

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1	ϵ	Z	\rightarrow	ϵ	1

Process *aabb*:

1 (0, *aabb*, Z)

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3 (0, *bb*, AAZ)

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Process *abb*:

1 (0, *abb*, Z)

2 (0, *bb*, AZ)

3 (1, *b*, Z)

4 (1, *b*, ϵ)

5 No rule applicable,
input not read entirely

PDA: important properties

- ▶ Γ and Σ do not need to be disjoint
 - ▶ Not uncommon: $\Sigma \subseteq \Gamma$
 - ▶ ... but convention: Σ lowercase letters, Γ uppercase letters
- ▶ ε transitions are possible
 - ▶ ... and can modify the stack
- ▶ PDAs as defined above are **non-deterministic**
 - ▶ Deterministic PDA: For each situation there is only one applicable transition (either ε or c)
 - ▶ deterministic PDAs are strictly weaker
- ▶ We can also define PDAs with acceptance via final states, but...
 - ▶ this makes representation of PDAs more complex
 - ▶ makes proofs more difficult

Define a PDA detecting all palindromes over $\Sigma = \{a, b\}$, i.e. all words

$$\{w \cdot \overleftarrow{w} \mid w \in \Sigma^*\}$$

where

$$\overleftarrow{w} = a_n \dots a_1 \text{ if } w = a_1 \dots a_n$$

Can you define a deterministic automaton?

Equivalence of PDAs and Context-Free Grammars

Theorem

The class of languages that can be accepted by a PDA is exactly the class of languages that can be produced by a context-free grammar.

Equivalence of PDAs and Context-Free Grammars

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The class of languages that can be accepted by a PDA is exactly the class of languages that can be produced by a context-free grammar.

Proof.

- ▶ For a cf. grammar G , generate a PDA \mathcal{A}_G with $L(\mathcal{A}_G) = L(G)$.
- ▶ For a PDA \mathcal{A} , generate a cf. grammar $G_{\mathcal{A}}$ with $L(G_{\mathcal{A}}) = L(\mathcal{A})$.



From context-free grammars to PDAs

For a grammar $G = (N, \Sigma, P, S)$, an equivalent PDA is:

$$\mathcal{A}_G = (\{q\}, \Sigma, \Sigma \cup N, \Delta, q, S)$$

$$\Delta = \{(q, \varepsilon, A, \gamma, q) \mid A \rightarrow \gamma \in P\} \cup \{(q, a, a, \varepsilon, q) \mid a \in \Sigma\}$$

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\mathcal{A}_G simulates the productions of G in the following way:

- ▶ a production rule is applied to the top stack symbol if it is an NTS
- ▶ a TS is removed from the stack if it corresponds to the next input character

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Note:

- ▶ \mathcal{A}_G is nondeterministic if there are several rules for one NTS.
- ▶ \mathcal{A}_G only has one single state.
 - ▶ Corollary: PDAs need no states, could be written as $(\Sigma, \Gamma, \Delta, Z_0)$.

From context-free grammars to PDAs: exercise

For the grammar $G = (\{S\}, \{a, b\}, P, S)$ with

$$P = \{S \rightarrow aSa$$

$$S \rightarrow bSb$$

$$S \rightarrow \varepsilon\}$$

- ▶ create an equivalent PDA \mathcal{A}_G ,
- ▶ show how \mathcal{A}_G processes the input $abba$.

From PDAs to context-free grammars

Transforming a PDA $\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, q_0, Z_0)$ into a grammar $G_{\mathcal{A}} = (N, \Sigma, P, S)$ is more involved:

- ▶ N contains symbols $[pZq]$, meaning
 - ▶ \mathcal{A} must go from p to q deleting Z from the stack
- ▶ for a transition $(p, a, Z, \varepsilon, q)$ that deletes a stack symbol:
 - ▶ \mathcal{A} can switch from p to q and delete Z by reading input a
 - ▶ this can be expressed by a production rule $[pZq] \rightarrow a$.
- ▶ for transitions (p, a, Z, ABC, q) that produce stack symbols:
 - ▶ test all possible transitions for removing these symbols
 - ▶ $[pZt] \rightarrow a[qAr][rBs][sCt]$ for all states r, s, t
 - ▶ intuitive meaning: in order to go from p to t and delete Z , you can
 - 1 read the input a
 - 2 go into state q
 - 3 find states r, s through which you can go from q to t and delete A, B , and C from the stack.

$G_{\mathcal{A}}$: formal definition

For $\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, q_0, Z_0)$ we define $G_{\mathcal{A}} = (N, \Sigma, P, S)$ as follows

- ▶ $N = \{S\} \cup \{[pZq] \mid p, q \in Q, Z \in \Gamma\}$
- ▶ P contains the following rules:
 - ▶ for every $q \in Q$, P contains $\{S \rightarrow [q_0Z_0q]\}$
meaning: \mathcal{A} has to go from q_0 to any state q , deleting Z_0 .
 - ▶ for each transition $(p, a, Z, Y_1Y_2 \dots Y_n, q)$ with
 - ▶ $a \in \Sigma \cup \{\varepsilon\}$ and
 - ▶ $Z, Y_1, Y_2 \dots Y_n \in \Gamma$,

$G_{\mathcal{A}}$: formal definition

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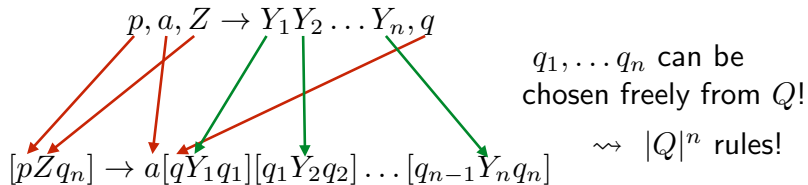
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 - ▶ $Z, Y_1, Y_2 \dots Y_n \in \Gamma$,

P contains rules

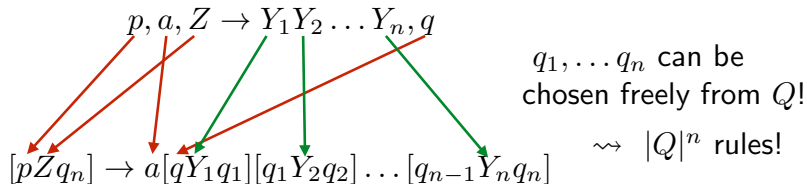
$$[pZq_n] \rightarrow a[qY_1q_1][q_1Y_2q_2] \dots [q_{n-1}Y_nq_n]$$

for all possible combinations of states $q_1, q_2, \dots, q_n \in Q$.

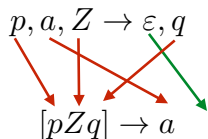
PDA to grammar illustrated



PDA to grammar illustrated

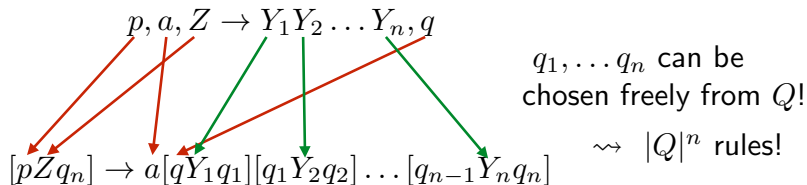


Special case: $n = 0$

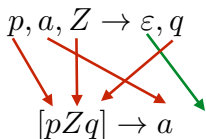


$\rightsquigarrow 1$ rule!

PDA to grammar illustrated

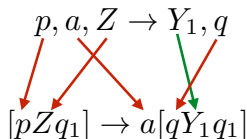


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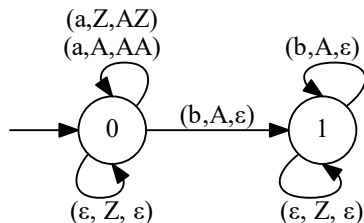


$\rightsquigarrow |Q|$ rules!

Exercise: transformation of PDA into grammar

$\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, 0, Z)$

- ▶ $Q = \{0, 1\}$
- ▶ $\Sigma = \{a, b\}$
- ▶ $\Gamma = \{A, Z\}$
- ▶ $\Delta :$



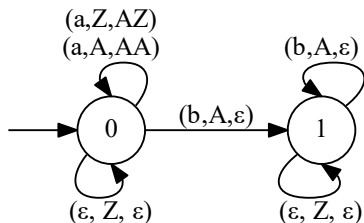
0	ε	Z	\rightarrow	ε	0
0	a	Z	\rightarrow	AZ	0
0	a	A	\rightarrow	AA	0
0	b	A	\rightarrow	ε	1
1	b	A	\rightarrow	ε	1
1	ε	Z	\rightarrow	ε	1

- ▶ Transform \mathcal{A} into a grammar $G_{\mathcal{A}}$ (and reduce $G_{\mathcal{A}}$).
- ▶ Show how \mathcal{A}_G produces the words ε , ab , and $aabb$.

Exercise: transformation of PDA into grammar

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Bonus Exercises/Homework

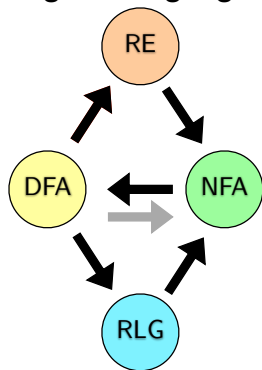
- ▶ Assume $\Sigma = \{a, b\}$.
 - ▶ Find a PDA A_1 that accepts $L_1 = \{w \in \Sigma^* \mid |w|_a = |w|_b\}$.
 - ▶ Give an accepting run of A_1 on *abbbaa*.
- ▶ Assume $\Sigma = \{a, b\}$.
 - ▶ Find a PDA A_2 that accepts $L_2 = \{w \in \Sigma^* \mid |w|_a < |w|_b\}$.
 - ▶ Give an accepting run of A_2 on *bbbaaaabb*.
- ▶ Assume $\Sigma = \{a, b, c\}$.
 - ▶ Find a PDA A_3 that accepts $L_3 = \{w \in \Sigma^* \mid |w| \text{ is odd and } w[(|w| + 1)/2] = a\}$ (the middle symbol is an *a*)
 - ▶ Give an accepting run of A_3 on *cccaabb*.
- ▶ Assume $\Sigma = \{a, b, c\}$.
 - ▶ Find a PDA A_4 that accepts $L_4 = \{a^n b^m c^o \mid n, m, o \in \mathbb{N}, n = m + o\}$
 - ▶ Give an accepting run of A_4 on *aacc*.

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 - ▶ Give an accepting run of A_4 on *aacc*.

Comparison: Regular vs. context-free languages

Regular languages

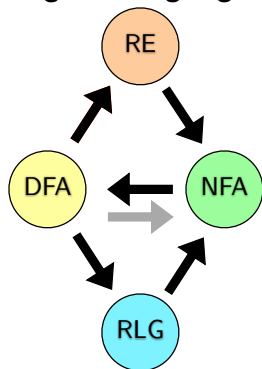


Context-free languages



Comparison: Regular vs. context-free languages

Regular languages



Context-free languages



For both language classes there are different but equivalent formal descriptions, supporting different arguments about the classes!

Outline

Introduction

Regular Languages and Finite Automata

Scanners and Flex

Formal Grammars and Context-Free Languages

Formal Grammars

The Chomsky Hierarchy

Right-linear Grammars

Context-free Grammars

Push-Down Automata

Properties of Context-free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

Beyond context-free languages

Theorem (Existence of non-context-free languages)

Let Σ be an alphabet. There are languages $L \subseteq \Sigma^$ such that there is no context-free grammar G with $L = L(G)$.*

- ▶ There are languages that are not context-free
 - ▶ e.g. $\{a^n b^n c^n \mid n \in \mathbb{N}\}$
 - ▶ ...but how do we show this?
- ▶ For regular languages: Pumping lemma
 - ▶ Finite automata must loop on long words
 - ▶ Loops can be repeated
 - ▶ Hence: A language that cannot be pumped is not regular
- ▶ For context-free languages?

Pumping-lemma for context-free languages

Idea:

- ▶ If a context-free grammar G can produce words of arbitrary length, there is least one **repeated NTS** in the derivation
 - ▶ If there are n rules in the grammar, at least one rule has to be used more than once in a derivation of length $n + 1$
 - ▶ Slightly stronger arguments (based on $|N|$ instead of $|P|$ are possible)
- ▶ Hence there is a derivation $A \Longrightarrow^* vAx$ for $v, x \in (\Sigma \cup \Gamma)^*$
 - ▶ ... and we can repeat this part of the derivation
 $A \Longrightarrow^* vAx \Longrightarrow^* vvAxx \Longrightarrow^* vvvAxxx$
 - ▶ If G has no chain rules, at least one of v, x is non-empty

Pumping Lemma I vs. Pumping Lemma II:

- ▶ PL I (regular languages)
 - ▶ Argument based on accepting automaton
 - ▶ Finite automaton must loop on sufficiently long word
 - ▶ One non-empty segment can be pumped
- ▶ PL II (context-free languages)
 - ▶ Argument based on generating grammar
 - ▶ At least one non-terminal must repeat in sufficiently long derivation
 - ▶ Two segments around this NTS can be pumped in parallel, at least one of which is non-empty

Pumping Lemma II

Theorem (Pumping-Lemma for context-free languages)

Let L be a language generated by a context-free grammar $G_L = (N, \Sigma, P, S)$ without contracting rules or chain rules. Let $m = |N|$, r be the maximum length of the rhs of a rule in P , and $k = r \cdot (m + 1)$.

Then:

For every $s \in L$ with $|s| > k$ there exists a segmentation

$u \cdot v \cdot w \cdot x \cdot y = s$ such that

- 1** $vx \neq \varepsilon$
- 2** $|vwx| \leq k$
- 3** $u \cdot v^h \cdot w \cdot x^h \cdot y \in L$ for every $h \in \mathbb{N}$.

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Let L be a context-free language. Then there exists $k \in \mathbb{N}$ with the following property:

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- 1 $vx \neq \varepsilon$
- 2 $|vwx| \leq k$
- 3 $u \cdot v^h \cdot w \cdot x^h \cdot y \in L$ for every $h \in \mathbb{N}$.

- ▶ Holds for $\{a^n b^n\}$, but not for $\{a^n b^n c^n\}$.
- ▶ $\{a^n b^n c^n\}$ is **not context-free**, but context-sensitive, as we have seen before.

Group Exercise: $a^n b^n c^n$

Use the Pumping Lemma II to show that $L = \{a^n b^n c^n \mid n \in \mathbb{N}\}$ is not context-free.

Closure properties

Theorem (Closure under $\cup, \cdot, *$)

The class of context-free languages is closed under union, concatenation, and Kleene star.

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The class of context-free languages is closed under union, concatenation, and Kleene star.

For context-free grammars

$$G_1 = (N_1, \Sigma, P_1, S_1) \quad \text{and} \quad G_2 = (N_2, \Sigma, P_2, S_2)$$

with $N_1 \cap N_2 = \{\}$ (rename NTSs if needed), let S be a new start symbol.

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For context-free grammars

$$G_1 = (N_1, \Sigma, P_1, S_1) \quad \text{and} \quad G_2 = (N_2, \Sigma, P_2, S_2)$$

with $N_1 \cap N_2 = \{\}$ (rename NTSs if needed), let S be a new start symbol.

- ▶ for $L(G_1) \cup L(G_2)$, add productions $S \rightarrow S_1, S \rightarrow S_2$.
- ▶ for $L(G_1) \cdot L(G_2)$, add production $S \rightarrow S_1 S_2$.
- ▶ for $L(G_1)^*$, add productions $S \rightarrow \varepsilon, S \rightarrow S_1 S$.

Theorem (Closure under \cap)

Context-free languages are not closed under intersection.

Theorem (Closure under \cap)

Context-free languages are not closed under intersection.

Otherwise, $\{a^n b^n c^n\}$ would be context-free:

- ▶ $\{a^n b^n c^m\}$ is context-free
- ▶ $\{a^m b^n c^n\}$ is context-free
- ▶ $\{a^n b^n c^n\} = \{a^n b^n c^m\} \cap \{a^m b^n c^n\}$

Exercise: closure properties

- 1 Define context-free grammars for $L_1 = \{a^n b^n c^m \mid n, m \geq 0\}$ and $L_2 = \{a^m b^n c^n \mid n, m \geq 0\}$.
- 2 Use L_1, L_2 and the known closure properties to show that context-free languages are not closed under complement.

Decision problems: word problem

Theorem (Word problem for cf. languages)

*For a word w and a context-free grammar G , it is **decidable** whether $w \in L(G)$ holds.*

Decision problems: word problem

Theorem (Word problem for cf. languages)

*For a word w and a context-free grammar G , it is **decidable** whether $w \in L(G)$ holds.*

Proof.

The CYK algorithm decides the word problem. □

Decision problems: emptiness problem

Theorem (Emptiness problem for cf. languages)

*For a context-free grammar G , it is **decidable** if $L(G) = \{\}$ holds.*

Decision problems: emptiness problem

Theorem (Emptiness problem for cf. languages)

For a context-free grammar G , it is *decidable* if $L(G) = \{\}$ holds.

Proof.

Let $G = (N, \Sigma, P, S)$.

Iteratively compute *productive* NTSs, i.e. symbols that produce terminal words as follows:

- 1 let $Z = \Sigma$
- 2 add all symbols A to Z for which there is a rule $A \rightarrow \beta$ with $\beta \in Z^*$
- 3 repeat step 2 until no further symbols can be added
- 4 $L(G) = \{\}$ iff $S \notin Z$.



Decision problems: equivalence problem

Theorem (Equivalence problem for cf. languages)

*For context-free grammars G_1, G_2 , it is **undecidable** if $L(G_1) = L(G_2)$ holds.*

Decision problems: equivalence problem

Theorem (Equivalence problem for cf. languages)

*For context-free grammars G_1, G_2 , it is **undecidable** if $L(G_1) = L(G_2)$ holds.*

This follows from undecidability of Post's Correspondence Problem.

- ▶ A PCP can be encoded in grammars such that the PCP has a solution if and only if the two grammars are equivalent
- ▶ Since the PCP is undecidable, so is the equivalence problem
- ▶ Detail? Who needs details?

Summary: context-free languages

- ▶ characterised by
 - ▶ context-free grammars
 - ▶ pushdown automata
- ▶ closure properties
 - ▶ closed under $\cup, *, \cdot$
 - ▶ not closed under $\cap, \bar{}$
- ▶ decision problems
 - ▶ decidable: $w \in L(G), L(G) = \{ \}$ (Chomsky NF, CYK algorithm)
 - ▶ undecidable: $L(G_1) = L(G_2)$
- ▶ can describe nested dependencies
 - ▶ structure of programming languages
 - ▶ natural language processing
- ▶ in compilers, these features are used by parsers (next chapter)

Outline

Introduction

Regular Languages and Finite Automata

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

Parsing: Motivation

Formal grammars describe formal languages

- ▶ A grammar has a set of rules
- ▶ Rules replace words with words
- ▶ A word that can be derived from the start symbol belongs to the language of the grammar

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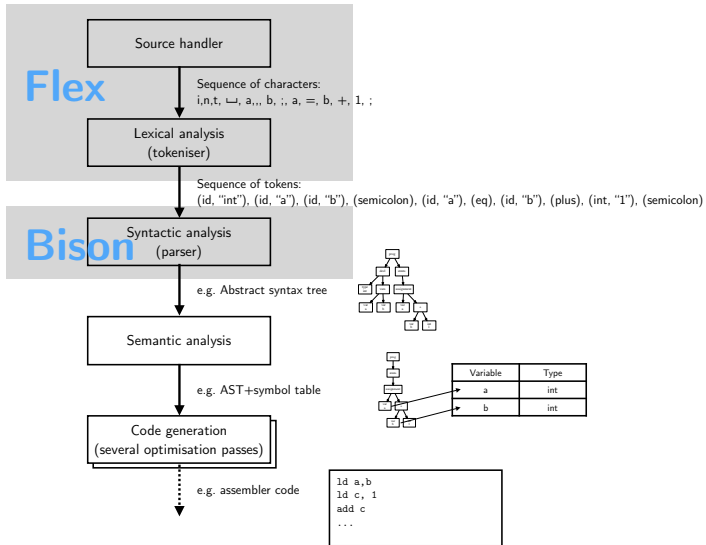
**In the concrete case of programming languages,
“words of the language” are syntactically correct programs!**

- ▶ Most programming languages are described by context-free grammars (with extra “semantic” constraints)
- ▶ Grammars **generate** languages
- ▶ PDAs and e.g. CYK-Parsing **recognize** words
- ▶ For compiler we need to ...
 - ▶ identify correct programs
 - ▶ and understand their structure!

Lexing and Parsing

- ▶ **Lexer:** Breaks programs into tokens
 - ▶ Smallest parts with semantic meaning
 - ▶ Can be recognized by regular languages/patterns
 - ▶ Example: `1`, `2`, `53` are all Integers
 - ▶ Example: `i`, `handle`, `stream` are all Identifiers
 - ▶ Example: `>`, `>=`, `*` are all individual operators
- ▶ **Parser:** Recognizes program structure
 - ▶ Language described by a grammar that has token types as terminals, not individual characters
 - ▶ Parser builds **parse tree**

Automatisation with Bison



- ▶ Yacc - Yet Another Compiler Compiler
 - ▶ Originally written \approx 1971 by Stephen C. Johnson at AT&T
 - ▶ LALR parser generator
 - ▶ Translates grammar into syntax analyser



YACC/Bison

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- ▶ GNU Bison
 - ▶ Written by Robert Corbett in 1988
 - ▶ Yacc-compatibility by Richard Stallman
 - ▶ Output languages now C, C++, Java
- ▶ Yacc, Bison, BYacc, . . . mostly compatible (POSIX P1003.2)

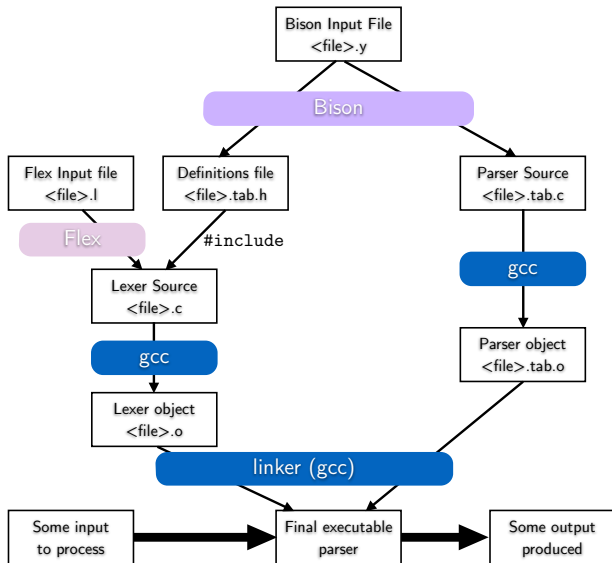
Yacc/Bison Background

- ▶ By default, Bison constructs a **1 token Look-Ahead Left-to-right Rightmost-derivation** or LALR(1) parser
 - ▶ Input tokens are processed **left-to-right**
 - ▶ Shift-reduce parser:
 - ▶ **Stack** holds tokens (terminals) and non-terminals
 - ▶ Tokens are **shifted** from input to stack. If the top of the stack contains symbols that represent the right hand side (RHS) of a grammar rule, the content is **reduced** to the LHS
 - ▶ Since input is reduced left-to-right, this corresponds to a **rightmost** derivation
 - ▶ Ambiguities are solved via look-ahead and special rules
 - ▶ If input can be reduced to start symbol, success!
 - ▶ Error otherwise
- ▶ LALR(1) is efficient in time and memory
 - ▶ Can parse “all reasonable languages”
 - ▶ For unreasonable languages, Bison (but not Yacc) can also construct **GLR** (General LR) parsers
 - ▶ Try all possibilities with back-tracking
 - ▶ Corresponds to the **non-determinism** of stack machines

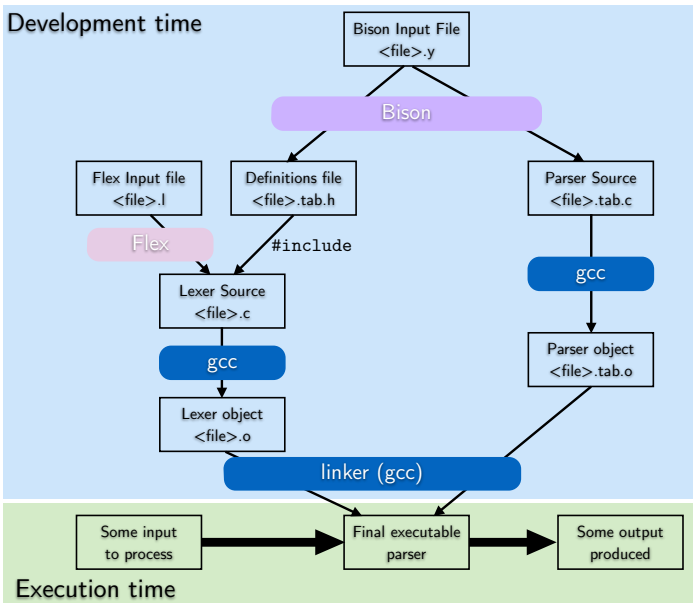
Yacc/Bison Overview

- ▶ Bison reads a specification file and converts it into (C) code of a parser
- ▶ Specification file: Declarations, grammar rules with actions, support code
 - ▶ Declarations: C declarations and data model, token names, associated values, includes
 - ▶ Grammar rules: Non-terminal with alternatives, **action** associated with each alternative
 - ▶ Support code: e.g. `main()` function, error handling...
 - ▶ Syntax similar to (F)lex
 - ▶ Sections separated by `%%`
 - ▶ Special commands start with `%`
- ▶ Bison generates function `yyparse()`
- ▶ Bison needs function `yylex()`
 - ▶ Usually provided via (F)lex

Yacc/Bison workflow



Yacc/Bison workflow



Example task: Desk calculator

- ▶ Desk calculator
 - ▶ Reads algebraic expressions and assignments
 - ▶ Prints result of expressions
 - ▶ Can store values in **registers** R0-R99

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[Shell] ./scicalc
R10=3*(5+4)
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R9=(3.1415*R10+3)
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```

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R9=(3.1415*R10+3)
> RegVal: 87.820500

R9+R10
> 114.820500

...
```

Abstract grammar for desk calculator (partial)

$$G_{DC} = (N, \Sigma, P, S)$$

- ▶ $\Sigma = \{\text{PLUS, MULT, ASSIGN, OPENPAR, CLOSEPAR, REGISTER, FLOAT, ...}\}$
 - ▶ Some terminals are single characters
(+, =, ...)
 - ▶ Others are complex
R10, 1.3e7
 - ▶ Terminals (“tokens”) are generated by the lexer
- ▶ $N = \{\text{stmt, assign, expr, term, factor, ...}\}$

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▶ $P:$

stmt	→	assign
		expr

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assign	→	REGISTER ASSIGN expr

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assign	→	REGISTER ASSIGN expr
expr	→	expr PLUS term
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term	→	term MULT factor
		factor
factor	→	REGISTER
		FLOAT
		OPENPAR expr CLOSEPAR

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		expr
assign	→	REGISTER ASSIGN expr
expr	→	expr PLUS term
		term
term	→	term MULT factor
		factor
factor	→	REGISTER
		FLOAT
		OPENPAR expr CLOSEPAR

▶ S :

- ▶ For a single statement, $S = \text{stmt}$
- ▶ In practice, we need to handle sequences of statements and empty input lines (not reflected in the grammar)

Parsing statements (1)

- ▶ Example string: $R10 = (4.5+3*7)$
- ▶ Tokenized: REGISTER ASSIGN OPENPAR FLOAT PLUS
FLOAT MULT FLOAT CLOSEPAR
 - ▶ In the following abbreviated R, A, O, F, P, F, M, F, C
- ▶ Parsing state:
 - ▶ Unread input (left column)
 - ▶ Current stack (middle column, top element on right)
 - ▶ How state was reached (right column)

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Input	Stack	Comment
R A O F P F M F C		Start

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Input	Stack	Comment
R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack

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O F P F M F C	R A	Shift A to stack

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P F M F C	R A O F	Shift F to stack

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A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F

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A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor

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F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor

...

Parsing statements (2)

R A O F P F M F C

A O F P F M F C

O F P F M F C

F P F M F C

P F M F C

P F M F C

P F M F C

R

R A

R A O

R A O F

R A O factor

R A O term

Start

Shift R to stack

Shift A to stack

Shift O to stack

Shift F to stack

Reduce F

Reduce factor

Parsing statements (2)

R A O F P F M F C	
A O F P F M F C	R
O F P F M F C	R A
F P F M F C	R A O
P F M F C	R A O F
P F M F C	R A O factor
P F M F C	R A O term
P F M F C	R A O expr

Start
Shift R to stack
Shift A to stack
Shift O to stack
Shift F to stack
Reduce F
Reduce factor
Reduce term

Parsing statements (2)

R A O F P F M F C	
A O F P F M F C	R
O F P F M F C	R A
F P F M F C	R A O
P F M F C	R A O F
P F M F C	R A O factor
P F M F C	R A O term
P F M F C	R A O expr
F M F C	R A O expr P

Start
Shift R to stack
Shift A to stack
Shift O to stack
Shift F to stack
Reduce F
Reduce factor
LA! Reduce term
Shift P

Parsing statements (2)

R A O F P F M F C

A O F P F M F C

O F P F M F C

F P F M F C

P F M F C

P F M F C

P F M F C

P F M F C

F M F C

M F C

R

R A

R A O

R A O F

R A O factor

R A O term

R A O expr

R A O expr P

R A O expr P F

Start

Shift R to stack

Shift A to stack

Shift O to stack

Shift F to stack

Reduce F

Reduce factor

LA! Reduce term

Shift P

Shift F

Parsing statements (2)

R A O F P F M F C	
A O F P F M F C	R
O F P F M F C	R A
F P F M F C	R A O
P F M F C	R A O F
P F M F C	R A O factor
P F M F C	R A O term
P F M F C	R A O expr
F M F C	R A O expr P
M F C	R A O expr P F
M F C	R A O expr P factor

Start
Shift R to stack
Shift A to stack
Shift O to stack
Shift F to stack
Reduce F
Reduce factor
LA! Reduce term
Shift P
Shift F
Reduce F

Parsing statements (2)

R A O F P F M F C

A O F P F M F C

O F P F M F C

F P F M F C

P F M F C

P F M F C

P F M F C

P F M F C

F M F C

M F C

M F C

M F C

R

R A

R A O

R A O F

R A O factor

R A O term

R A O expr

R A O expr P

R A O expr P F

R A O expr P factor

R A O expr P term

Start

Shift R to stack

Shift A to stack

Shift O to stack

Shift F to stack

Reduce F

Reduce factor

LA! Reduce term

Shift P

Shift F

Reduce F

Reduce factor

Parsing statements (2)

R A O F P F M F C	
A O F P F M F C	R
O F P F M F C	R A
F P F M F C	R A O
P F M F C	R A O F
P F M F C	R A O factor
P F M F C	R A O term
P F M F C	R A O expr
F M F C	R A O expr P
M F C	R A O expr P F
M F C	R A O expr P factor
M F C	R A O expr P term
F C	R A O expr P term M

Start
Shift R to stack
Shift A to stack
Shift O to stack
Shift F to stack
Reduce F
Reduce factor
LA! Reduce term
Shift P
Shift F
Reduce F
Reduce factor
LA! Shift M

Parsing statements (2)

R A O F P F M F C	
A O F P F M F C	R
O F P F M F C	R A
F P F M F C	R A O
P F M F C	R A O F
P F M F C	R A O factor
P F M F C	R A O term
P F M F C	R A O expr
F M F C	R A O expr P
M F C	R A O expr P F
M F C	R A O expr P factor
M F C	R A O expr P term
F C	R A O expr P term M
C	R A O expr P term M F

Start
Shift R to stack
Shift A to stack
Shift O to stack
Shift F to stack
Reduce F
Reduce factor
LA! Reduce term
Shift P
Shift F
Reduce F
Reduce factor
LA! Shift M
Shift F

Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F

Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf

Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
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P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt

Parsing statements (2)

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A O F P F M F C	R	Shift R to stack
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P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt
	R A O expr C	Shift C

Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
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P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt
	R A O expr C	Shift C
	R A factor	Reduce OeC

Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt
	R A O expr C	Shift C
	R A factor	Reduce OeC
	R A term	Reduce factor

Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt
	R A O expr C	Shift C
	R A factor	Reduce OeC
	R A term	Reduce factor
	R A expr	Reduce term

Parsing statements (2)

R A O F P F M F C

A O F P F M F C

O F P F M F C

F P F M F C

P F M F C

P F M F C

P F M F C

P F M F C

F M F C

M F C

M F C

M F C

F C

C

C

C

C

R

R A

R A O

R A O F

R A O factor

R A O term

R A O expr

R A O expr P

R A O expr P F

R A O expr P factor

R A O expr P term

R A O expr P term M

R A O expr P term M F

R A O expr P term M factor

R A O expr P term

R A O expr

R A O expr C

R A factor

R A term

R A expr

assign

Start

Shift R to stack

Shift A to stack

Shift O to stack

Shift F to stack

Reduce F

Reduce factor

LA! Reduce term

Shift P

Shift F

Reduce F

Reduce factor

LA! Shift M

Shift F

Reduce F

Reduce tMf

Reduce ePt

Shift C

Reduce OeC

Reduce factor

Reduce term

Reduce RAe

Parsing statements (2)

R A O F P F M F C

A O F P F M F C

O F P F M F C

F P F M F C

P F M F C

P F M F C

P F M F C

P F M F C

F M F C

M F C

M F C

M F C

F C

C

C

C

C

R

R A

R A O

R A O F

R A O factor

R A O term

R A O expr

R A O expr P

R A O expr P F

R A O expr P factor

R A O expr P term

R A O expr P term M

R A O expr P term M F

R A O expr P term M factor

R A O expr P term

R A O expr

R A O expr C

R A factor

R A term

R A expr

assign

stmt

Start

Shift R to stack

Shift A to stack

Shift O to stack

Shift F to stack

Reduce F

Reduce factor

LA! Reduce term

Shift P

Shift F

Reduce F

Reduce factor

LA! Shift M

Shift F

Reduce F

Reduce tMf

Reduce ePt

Shift C

Reduce OeC

Reduce factor

Reduce term

Reduce RAe

Reduce assign

Lexer interface

- ▶ Bison parser requires `yylex()` function
- ▶ `yylex()` returns **token**
 - ▶ Token text is defined by regular expression pattern
 - ▶ Tokens are encoded as integers
 - ▶ Symbolic names for tokens are defined by Bison in generated header file
 - ▶ By convention: Token names are all `CAPITALS`
- ▶ `yylex()` provides optional **semantic value** of token
 - ▶ Stored in global variable `yyval`
 - ▶ Type of `yyval` defined by Bison in generated header file
 - ▶ Default is `int`
 - ▶ For more complex situations often a `union`
 - ▶ For our example: Union of `double` (for floating point values) and `integer` (for register numbers)

Lexer for desk calculator (1)

```
/*  
Lexer for a minimal "scientific" calculator.  
  
Copyright 2014 by Stephan Schulz, schulz@eprover.org.  
  
This code is released under the GNU General Public Licence  
Version 2.  
*/  
  
%option noyywrap  
  
%{  
    #include "scicalcparse.tab.h"  
%}
```

Lexer for desk calculator (2)

```
DIGIT      [0-9]
INT        {DIGIT}+
PLAINFLOAT {INT} | {INT} [.] | {INT} [.] {INT} | [.] {INT}
EXP        [eE] (\+|-)?{INT}
NUMBER     {PLAINFLOAT}{EXP}?
REG        R{DIGIT}{DIGIT}?
```

```
%%
```

```
"*" {return MULT;}
"+" {return PLUS;}
"=" {return ASSIGN;}
"(" {return OPENPAR;}
")" {return CLOSEPAR;}
\n  {return NEWLINE;}
```

Lexer for desk calculator (3)

```
{REG}    {
    yylval.regno = atoi(yytext+1);
    return REGISTER;
}

{NUMBER} {
    yylval.val = atof(yytext);
    return FLOAT;
}

[ ] { /* Skip whitespace*/ }

/* Everything else is an invalid character. */
.    { return ERROR;}

%%
```

Data model of desk calculator

- ▶ Desk calculator has simple state
 - ▶ 100 floating point registers
 - ▶ R0-R99
- ▶ Represented in C as array of doubles:

```
#define MAXREGS 100
```

```
double regfile[MAXREGS];
```

- ▶ Needs to be initialized in support code

Bison code for desk calculator: Declarations

```
%{
#include <stdio.h>

#define MAXREGS 100

double regfile[MAXREGS];

extern int yyerror(char* err);
extern int yylex(void);
}%

%union {
double val;
int    regno;
}
```

Bison code for desk calculator: Tokens

```
%start stmtseq
```

```
%left PLUS
```

```
%left MULT
```

```
%token ASSIGN
```

```
%token OPENPAR
```

```
%token CLOSEPAR
```

```
%token NEWLINE
```

```
%token REGISTER
```

```
%token FLOAT
```

```
%token ERROR
```

```
%%
```

Actions in Bison

- ▶ Bison is based on syntax rules with associated actions
 - ▶ Whenever a **reduce** is performed, the action associated with the rule is executed
- ▶ Actions can be arbitrary C code
- ▶ Frequent: **semantic actions**
 - ▶ The action sets a **semantic value** based on the semantic value of the symbols reduced by the rule
 - ▶ For terminal symbols: Semantic value is `yylval` from Flex
 - ▶ Semantic actions have “historically valuable” syntax
 - ▶ Value of reduced symbol: `$$`
 - ▶ Value of first symbol in syntax rule body: `$1`
 - ▶ Value of second symbol in syntax rule body: `$2`
 - ▶ ...
 - ▶ Access to named components: `$<val>1`

Bison code for desk calculator: Grammar (1)

```
stmtseq: /* Empty */
        | NEWLINE stmtseq      {}
        | stmt  NEWLINE stmtseq {}
        | error NEWLINE stmtseq {}; /* After an error,
start afresh */
```

- ▶ Head: sequence of statements
- ▶ First body line: Skip empty lines
- ▶ Second body line: separate current statement from rest
- ▶ Third body line: After parse error, start again with new line

Bison code for desk calculator: Grammar (2)

```
stmt: assign {printf("> RegVal: %f\n", $<val>1);}
      |expr   {printf("> %f\n", $<val>1)};

assign: REGISTER ASSIGN expr {regfile[$<regno>1] = $<val>3;
      $<val>$ = $<val>3;} ;

expr: expr PLUS term {$<val>$ = $<val>1 + $<val>3;}
      | term {$<val>$ = $<val>1};

term: term MULT factor {$<val>$ = $<val>1 * $<val>3;}
      | factor {$<val>$ = $<val>1};

factor: REGISTER {$<val>$ = regfile[$<regno>1];}
        | FLOAT {$<val>$ = $<val>1;}
        | OPENPAR expr CLOSEPAR {$<val>$ = $<val>2};
```

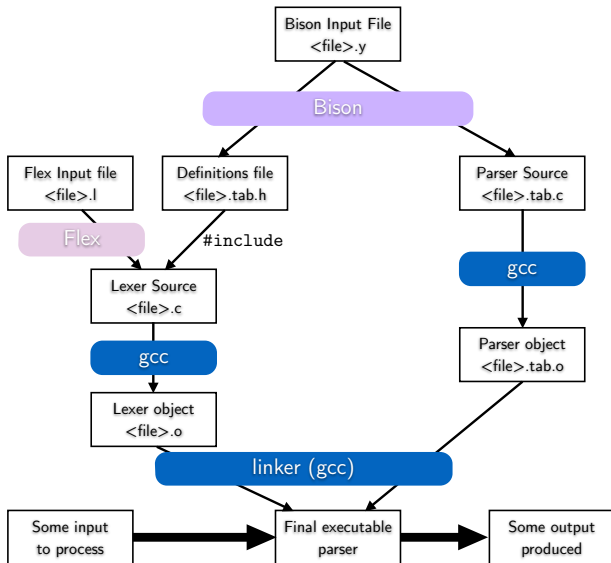
Bison code for desk calculator: Support code

```
int yyerror(char* err)
{
    printf("Error: %s\n", err);
    return 0;
}

int main (int argc, char* argv[])
{
    int i;

    for(i=0; i<MAXREGS; i++)
    {
        regfile[i] = 0.0;
    }
    return yyparse();
}
```

Bison workflow and dependencies



Building the calculator

1 Generate parser C code and include file for lexer

- ▶ `bison -d scicalcparse.y`
- ▶ Generates `scicalcparse.tab.c` and `scicalcparse.tab.h`

2 Generate lexer C code

- ▶ `flex -t scicalclex.l > scicalclex.c`

3 Compile lexer

- ▶ `gcc -c -o scicalclex.o scicalclex.c`

4 Compile parser and support code

- ▶ `gcc -c -o scicalcparse.tab.o scicalcparse.tab.c`

5 Link everything

- ▶ `gcc scicalclex.o scicalcparse.tab.o -o scicalc`

6 Fun!

- ▶ `./scicalc`

Exercise: calculator

▶ Exercise 1:

- ▶ Go to `http://wwwlehre.dhbw-stuttgart.de/~sschulz/fla2018.html`
- ▶ Download `scicalcparse.y` and `scicalclex.l`
- ▶ Build the calculator
- ▶ Run and test the calculator

▶ Exercise 2:

- ▶ Add support for division and subtraction `/`, `-`
- ▶ Add support for unary minus (the negation operator `-`)

Definition (derivation)

For a grammar G , a **derivation** of a word w_n in $L(G)$ is a sequence $S \Rightarrow w_1 \Rightarrow \dots \Rightarrow w_n$ where S is the start symbol, and each w_i is generated from its predecessor by application of a production of the grammar

Example: derivation

Example (well-formed expressions over $x, +, *, (,)$)

Let $G_E = (N, \Sigma, P, E)$ with

- ▶ $N = \{E\}$
- ▶ $\Sigma = \{(,), +, *, x\}$
- ▶ P :
 - 1 $E \rightarrow x$
 - 2 $E \rightarrow (E)$
 - 3 $E \rightarrow E + E$
 - 4 $E \rightarrow E * E$

Example: derivation

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The following is a derivation of $x + x + x * x$ (with the replaced symbol printed bold):

E

Example: derivation

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The following is a derivation of $x + x + x * x$ (with the replaced symbol printed bold):

$$\begin{array}{l} E \\ \Rightarrow \mathbf{E} + E \end{array}$$

Example: derivation

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Example: derivation

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Rightmost and leftmost Derivations

Definition (rightmost/leftmost)

- ▶ A derivation is called a **rightmost** derivation, if at any step it replaces the **rightmost** non-terminal in the current word.
- ▶ A derivation is called a **leftmost** derivation, if at any step it replaces the **leftmost** non-terminal in the current word.

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Example

- ▶ The derivation on the previous slide is neither leftmost nor rightmost.
- ▶ A **rightmost derivation** is:

$$\mathbf{E} \Rightarrow E + \mathbf{E} \Rightarrow E + E + \mathbf{E} \Rightarrow E + E + E * \mathbf{E} \Rightarrow E + E + \mathbf{E} * x \Rightarrow \\ E + \mathbf{E} + x * x \Rightarrow \mathbf{E} + x + x * x \Rightarrow x + x + x * x$$

Parse trees

Definition (parse tree)

A **parse tree** for a derivation in a grammar $G = (N, \Sigma, P, S)$ is an ordered, labelled tree with the following properties:

- ▶ Each node is labelled with a symbol from $N \cup \Sigma$
- ▶ The root of the tree is labelled with the start symbol S .
- ▶ Each inner node is labelled with a single non-terminal symbol from V_N
- ▶ If an inner node with label A has children labelled with symbols $\alpha_1, \dots, \alpha_n$, then there is a production $A \rightarrow \alpha_1 \dots \alpha_n$ in P .

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-
- ▶ The parse tree represents a derivation of the word formed by the labels of the leaf nodes
 - ▶ It abstracts from the order in which productions are applied.

Example: parse trees

Example

The derivation $\mathbf{E} \Rightarrow E + \mathbf{E} \Rightarrow E + E + \mathbf{E} \Rightarrow E + E + E * \mathbf{E} \Rightarrow$
 $E + E + \mathbf{E} * x \Rightarrow E + \mathbf{E} + x * x \Rightarrow \mathbf{E} + x + x * x \Rightarrow x + x + x * x$
can be represented by a sequence of parse trees:

Example: parse trees

Example

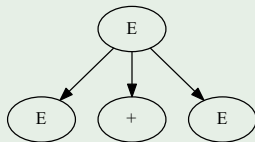
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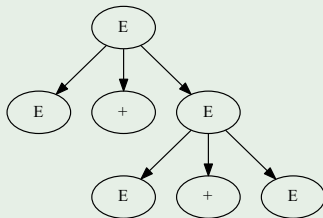
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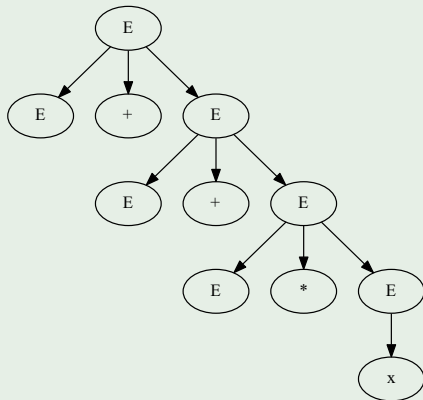
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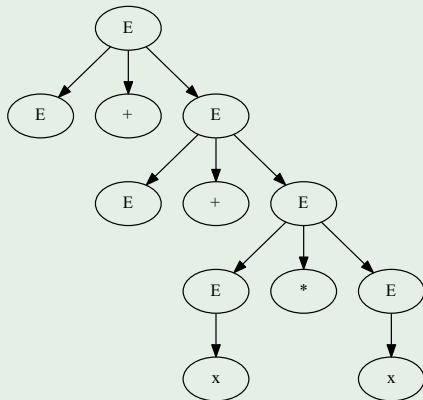
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Example: parse trees

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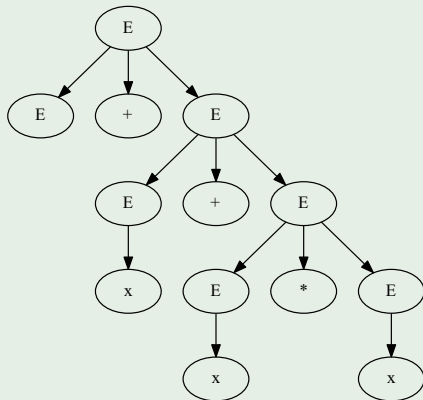
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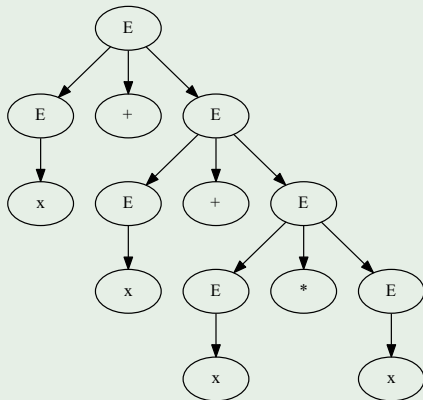
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Example: parse trees

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Ambiguity

Definition (ambiguous)

A grammar $G = (N, \Sigma, P, S)$ is **ambiguous** if it has multiple different parse trees for a word w in $L(G)$.

Ambiguity

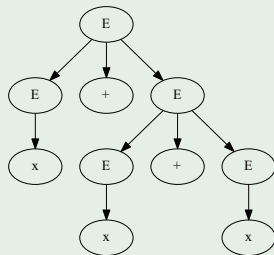
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G_E is ambiguous since it has several parse trees for $x + x + x$.

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Ambiguity

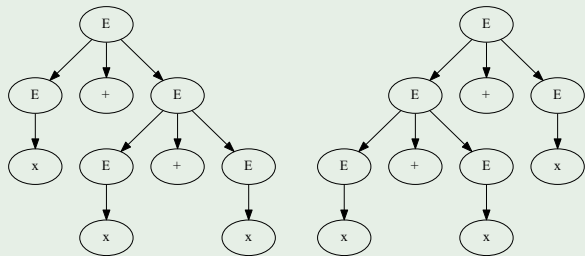
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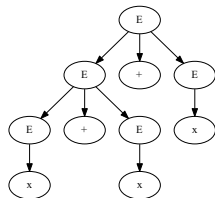
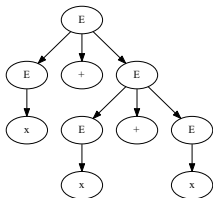
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- 3 $E \rightarrow E + E$
- 4 $E \rightarrow E * E$



Exercise: Ambiguity is worse...

Consider the grammar G_E and its parse trees for $x + x + x$.

- 1 $E \rightarrow x$
- 2 $E \rightarrow (E)$
- 3 $E \rightarrow E + E$
- 4 $E \rightarrow E * E$



- ▶ Provide a rightmost derivation for the right tree.
- ▶ Provide a rightmost derivation for the left tree.
- ▶ Provide a leftmost derivation for the left tree.
- ▶ Provide a leftmost derivation for the right tree.

Exercise: Eliminating Ambiguity

Consider G_E with the following productions:

1 $E \rightarrow x$

2 $E \rightarrow (E)$

3 $E \rightarrow E + E$

4 $E \rightarrow E * E$

Define a grammar G' with $L(G) = L(G')$ that is not ambiguous, that respects that $*$ has a higher precedence than $+$, and that respects left-associativity for all operators.

Abstract Syntax Trees

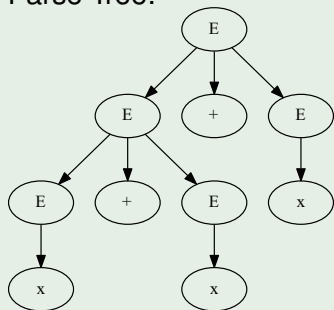
- ▶ **Abstract Syntax Trees** represent the structure of a derivation without the specific details
- ▶ Think: “Parse trees without redundant syntactic elements”

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Example

Parse Tree:

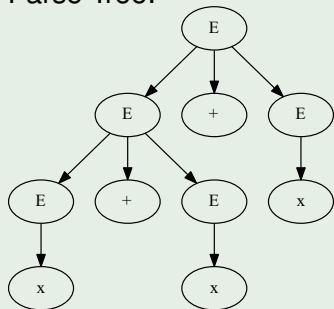


Abstract Syntax Trees

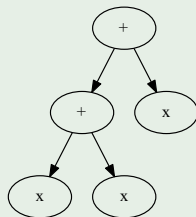
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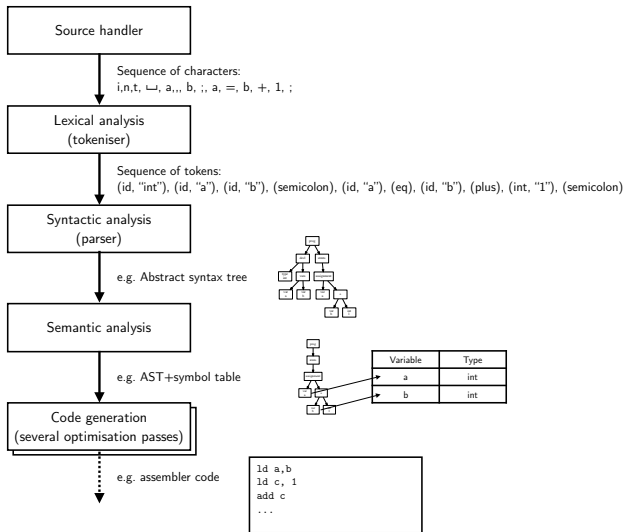
Parse Tree:



Corresponding AST:



High-Level Architecture of a Compiler



Syntactic Analysis/Parsing

- ▶ Description of the language with a **context-free grammar**
- ▶ Parsing:
 - ▶ Try to build a **parse tree**/abstract syntax tree (AST)
 - ▶ Parse tree unambiguously describes structure of a program
 - ▶ AST reflects abstract syntax (can e.g. drop parenthesis)
- ▶ Methods:
 - ▶ Manual recursive descent parser
 - ▶ Automatic with a table-driven bottom-up parser

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Result: Abstract Syntax Tree

Semantic Analysis

- ▶ Analyze static properties of the program
 - ▶ Which variable has which type?
 - ▶ Are all expressions well-typed?
 - ▶ Which names are defined?
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- ▶ Core tool: Symbol table

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Result: Annotated AST

- ▶ Transform Abstract Syntax Tree to generate better code
 - ▶ Smaller
 - ▶ Faster
 - ▶ Both
- ▶ Mechanisms
 - ▶ Common sub-expression elimination
 - ▶ Loop unrolling
 - ▶ Dead code/data elimination
 - ▶ ...

Optimization

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 - ▶ Smaller
 - ▶ Faster
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Result: Optimized AST

- ▶ Convert optimized AST into low-level code
- ▶ Target languages:
 - ▶ Assembly code
 - ▶ Machine code
 - ▶ VM code (z.B. JAVA byte-code, p-Code)
 - ▶ C (as a “portable assembler”)
 - ▶ ...

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Result: Program in target language

Summary: parsers

Parsers

- ▶ recognise structure of programs
- ▶ receive **tokens** from scanner
- ▶ construct **parse tree** and symbol table
- ▶ common: **shift-reduce** parsing

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- ▶ uses grammar rules to perform semantic actions
- ▶ uses look-ahead to resolve conflicts

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End lecture 16

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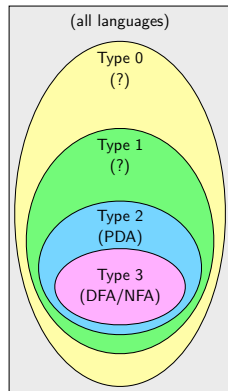
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Turing Machines

Turing Machine: Motivation

Four classes of languages described by grammars and equivalent machine models:

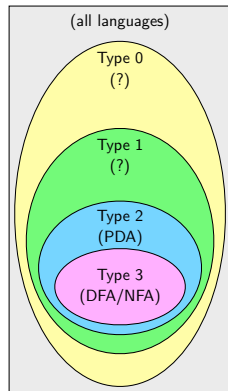
- 1 regular languages \rightsquigarrow finite automata
- 2 context-free languages \rightsquigarrow pushdown automata
- 3 context-sensitive languages \rightsquigarrow ?
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We need a machine model that is more powerful than PDAs:

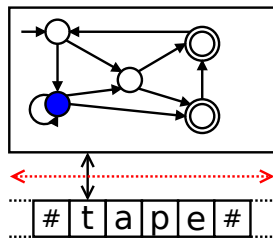
Turing Machines

Turing Machine: history

- ▶ proposed in 1936 by Alan Turing
 - ▶ paper: *On computable numbers, with an application to the Entscheidungsproblem*
 - ▶ uses the TM to show that satisfiability of first-order formulas is **undecidable**
- ▶ model of a **universal computer**
 - ▶ very simple (and thus easy to describe formally)
 - ▶ but as powerful as any conceivable machine



Turing Machine: conceptual model



- ▶ medium: unlimited **tape** (bidirectional)
 - ▶ initially contains input (and blanks #)
 - ▶ TM can read and **write** tape
 - ▶ TM can **move arbitrarily** over tape
 - ▶ serves for input, working, output
 - ▶ **output** possible
- ▶ transition relation
 - ▶ read and write current position
 - ▶ moving instructions (l, r, n)
- ▶ acceptance condition
 - ▶ **final state** is reached
 - ▶ no transitions possible
- ▶ commonalities with FA
 - ▶ control unit (finite set of states),
 - ▶ initial and final states
 - ▶ input alphabet

Transitions in Turing Machines

$$\Delta \subseteq Q \times \Gamma \times \Gamma \times \{l, n, r\} \times Q$$

- ▶ TM is in state
- ▶ reads tape symbol from current position
- ▶ writes tape symbol on current position
- ▶ moves to left, right, or stays
- ▶ goes into a new state

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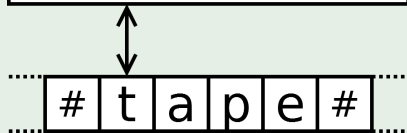
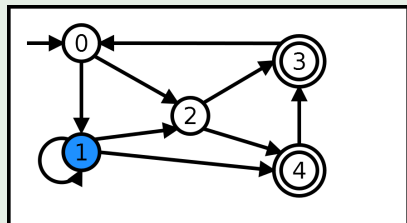
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A transition p, a, b, l, q can also be written as

$$p \ a \ \rightarrow \ b \ l \ q$$

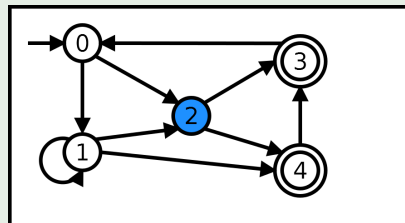
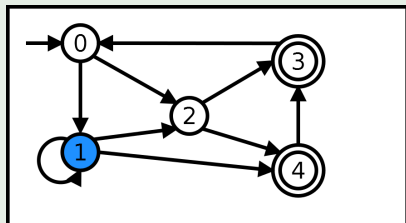
Example: transition

Example (transition $1, t \rightarrow c, r, 2$)



Example: transition

Example (transition $1, t \rightarrow c, r, 2$)



Definition (Turing Machine)

A **Turing Machine** (TM) is a 6-tuple $(Q, \Sigma, \Gamma, \Delta, q_0, F)$ where

- ▶ Q, Σ, q_0, F are defined as for NFAs,
- ▶ $\Gamma \supseteq \Sigma \cup \{\#\}$ is the **tape alphabet**, including at least Σ and the blank symbol,
- ▶ $\Delta \subseteq Q \times \Gamma \times \Gamma \times \{l, n, r\} \times Q$ is the transition relation.

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Note:

- ▶ Γ (the tape alphabet) can contain additional characters
- ▶ This does not increase the power, but makes finding TMs easier

Definition (configuration)

A **configuration** $c = \alpha q \beta$ of a Turing Machine is given by

- ▶ the current state q
- ▶ the tape content α on the left of the read/write head (except unlimited # sequences)
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Configurations of TMs

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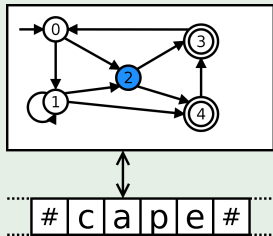
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A configuration c is a **stop configuration** if there are no transitions from c .

Example: configuration

Example: configuration

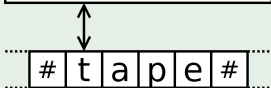
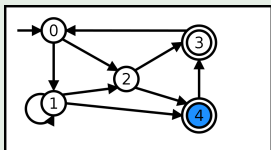
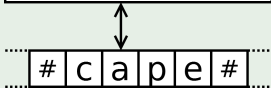
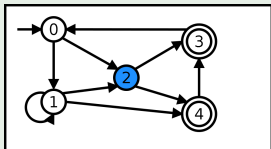
Example (configurations)



- ▶ This TM is in the configuration $c2ape$.

Example: configuration

Example (configurations)



▶ This TM is in the configuration $c2ape$.

▶ The configuration $4tape$ is accepting.

▶ If there are no transitions $4, t \rightarrow \dots$, $4tape$ also is a stop configuration.

Definition (computation, acceptance)

A **computation** of a TM \mathcal{M} on a word w is a sequence of configurations (according to the transition function) of configurations of \mathcal{M} , starting from q_0w .

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\mathcal{M} **accepts** w if there exists a computation of \mathcal{M} on w that leads to an accepting stop configuration.

Exercise: Turing Machines

Let $\Sigma = \{a, b\}$ and $L = \{w \in \Sigma^* \mid |w|_a \text{ is even}\}$.

- ▶ Give a TM \mathcal{M} that accepts (exactly) the words in L .
- ▶ Give the computation of \mathcal{M} on the words *abbab* and *bbab*.

Example: TM for $a^n b^n c^n$

$\mathcal{M} = (Q, \Sigma, \Gamma, \Delta, \text{start}, \{f\})$ with

- ▶ $Q = \{\text{start, findb, findc, check, back, end, f}\}$
- ▶ $\Sigma = \{a, b, c\}$ and $\Gamma = \Sigma \cup \{\#, x, y, z\}$

state	read	write	move	state	state	read	write	move	state
start	#	#	n	f	back	z	z	l	back
start	a	x	r	findb	back	b	b	l	back
findb	a	a	r	findb	back	y	y	l	back
findb	y	y	r	findb	back	a	a	l	back
findb	b	y	r	findc	back	x	x	r	start
findc	b	b	r	findc	end	z	z	l	end
findc	z	z	r	findc	end	y	y	l	end
findc	c	z	r	check	end	x	x	l	end
check	c	c	l	back	end	#	#	n	f
check	#	#	l	end					

Exercise: Turing Machines (2)

- a) Simulate the computations of \mathcal{M} on $aabbcc$ and $aabc$.
- b) Develop a Turing Machine \mathcal{P} accepting $L_{\mathcal{P}} = \{w cw \mid w \in \{a, b\}^*\}$.
- c) How do you have to modify \mathcal{P} if you want to recognise inputs of the form ww ?

Turing Machines with several tapes

- ▶ A k -tape TM has k tapes on which the heads can move independently.
- ▶ $\Delta \subseteq Q \times \Gamma^k \times \Gamma^k \times \{r, l, n\}^k \times Q$
- ▶ It is possible to simulate a k -tape TM with a (1-tape) TM:
 - ▶ use alphabet $\Gamma^k \times \{X, \#\}^k$
 - ▶ the first k language elements encode the tape content, the remaining ones the positions of the heads.

Reminder

- ▶ just like FAs and PDAs, TMs can be deterministic or non-deterministic, depending on the transition relation.
- ▶ for non-deterministic TMs, the machine accepts w if there **exists** a sequence of transitions leading to an accepting stop configuration.

Simulating non-deterministic TMs

Theorem (equivalence of deterministic and non-deterministic TMs)

Deterministic TMs can simulate computations of non-deterministic TMs; i.e. they describe the same class of languages.

Simulating non-deterministic TMs

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Proof.

Use a 3-tape TM:

- ▶ tape 1 stores the input w
- ▶ tape 2 enumerates all possible sequences of non-deterministic choices (for all non-deterministic transitions)
- ▶ tape 3 encodes the computation on w with choices stored on tape 2.



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Theorem (equivalence of TMs and unrestricted grammars)

The class of languages that can be accepted by a Turing Machine is exactly the class of languages that can be generated by unrestricted Chomsky grammars.

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The class of languages that can be accepted by a Turing Machine is exactly the class of languages that can be generated by unrestricted Chomsky grammars.

Proof.

- 1 simulate grammar derivations with a TM
- 2 simulate a TM computation with a grammar



Simulating a Type-0-grammar G with a TM

Use a non-deterministic 2-tape TM:

- ▶ tape 1 stores input word w
- ▶ tape 2 simulates the derivations of G , starting with S
 - ▶ (non-deterministically) choose a position
 - ▶ if the word starting at the position, matches α of a rule $\alpha \rightarrow \beta$, apply the rule
 - ▶ move tape content if necessary
 - ▶ replace α with β
 - ▶ compare content of tape 2 with tape 1
 - ▶ if they are equal, accept
 - ▶ otherwise continue

Simulating a TM with a Type-0-grammar

Goal: transform TM $\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, q_0, F)$ into grammar G

Technical difficulty:

- ▶ \mathcal{A} receives word as input **at the start**, possibly modifies it, then possibly accepts.
- ▶ G starts with S , applies rules, possibly generating w **at the end**.

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Technical difficulty:

- ▶ \mathcal{A} receives word as input **at the start**, possibly modifies it, then possibly accepts.
 - ▶ G starts with S , applies rules, possibly generating w **at the end**.
- 1** generate initial configuration $q_0w \in \Sigma^*$ with blanks left and right
 - 2** simulate the computation of \mathcal{A} on w

$$(p, a, b, r, q) \rightsquigarrow pa \rightarrow bq$$

$$(p, a, b, l, q) \rightsquigarrow cpa \rightarrow qcb \text{ (for all } c \in \Gamma)$$

$$(p, a, b, n, q) \rightsquigarrow pa \rightarrow qb$$

- 3** if an accepting stop configuration is reached, recreate w
 - ▶ requires a more complex alphabet with a “backup track”

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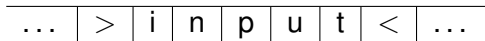
Selected Solutions

Linear bounded automata

- ▶ context-sensitive grammars do not allow for contracting rules
- ▶ a **linear bounded automaton (LBA)** is a TM that only uses the space originally occupied by the input w .
- ▶ limits of w are indicated by markers that cannot be passed by the read/write head

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Equivalence of cs. grammars and LBAs

Transformation of cs. grammar G into LBA:

- ▶ as for Type-0-grammar: use 2-tape-TM
 - ▶ input on tape 1
 - ▶ simulate operations of G on tape 2
- ▶ since the productions of G are non-contracting, words longer than w need not be considered

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Transformation of LBA \mathcal{A} into cs. grammar:

- ▶ similar to construction for TM:
 - ▶ generate w **without blanks**
 - ▶ simulate operation of \mathcal{A} on w
 - ▶ rules are non-contracting ✓

Closure properties: regular operations

Theorem (closure under $\cup, \cdot, *$)

*The class of languages described by context-sensitive grammars is closed under $\cup, \cdot, *$.*

Closure properties: regular operations

Theorem (closure under $\cup, \cdot, *$)

*The class of languages described by context-sensitive grammars is closed under $\cup, \cdot, *$.*

Proof.

Concatenation and Kleene-star are more complex than for cf. grammars because the context can influence rule applicability.

- ▶ rename NTSs (as for cf. grammars)
- ▶ only allow NTSs as context
- ▶ only allow productions of the kind
 - ▶ $N_1N_2 \dots N_k \rightarrow M_1M_2 \dots M_j$
 - ▶ $N \rightarrow a$



Closure properties: intersection and complement

Theorem (closure under \cap)

The class of context-sensitive languages is closed under intersection.

Closure properties: intersection and complement

Theorem (closure under \cap)

The class of context-sensitive languages is closed under intersection.

Proof.

- ▶ use a 2-tape-LBA
- ▶ simulate computation of \mathcal{A}_1 on tape 1, \mathcal{A}_2 on tape 2
- ▶ accept if both \mathcal{A}_1 and \mathcal{A}_2 accept



Closure properties: intersection and complement

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Theorem (closure under $\overline{}$)

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Theorem (closure under $\overline{}$)

The class of context-sensitive languages is closed under complement.

- ▶ shown in 1988

Theorem (Word problem for cs. languages)

*The **word** problem for cs. languages is **decidable**.*

Theorem (Word problem for cs. languages)

The *word* problem for cs. languages is *decidable*.

Proof.

- ▶ N , Σ and P are finite
- ▶ rules are non-contracting
- ▶ for a word of length n only a finite number of derivations up to length n has to be considered.



Context-sensitive grammars: decision problems (cont')

Theorem (Emptiness problem for cs. languages)

*The **emptiness** problem for cs. languages is **undecidable**.*

Proof.

Also follows from undecidability of Post's correspondence problem. □

Context-sensitive grammars: decision problems (cont')

Theorem (Emptiness problem for cs. languages)

The *emptiness* problem for cs. languages is *undecidable*.

Proof.

Also follows from undecidability of Post's correspondence problem. □

Theorem (Equivalence problem for cs. languages)

The *equivalence* problem for cs. languages is *undecidable*.

Proof.

If this problem was decidable for cs. languages, it would also be decidable for cf. languages (since every cf. language is also cs.). □

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The universal Turing Machine \mathcal{U}

- ▶ \mathcal{U} is a TM that simulates other Turing Machines
- ▶ since TMs have finite alphabets and state sets, they can be encoded by a (binary) alphabet by an encoding function $c()$
- ▶ Input:
 - ▶ encoding $c(\mathcal{A})$ of a TM \mathcal{A} on tape 1
 - ▶ encoding $c(w)$ of an input word w for \mathcal{A} on tape 2
- ▶ with input $c(\mathcal{A})$ and $c(w)$, \mathcal{U} behaves exactly like \mathcal{A} on w :
 - ▶ \mathcal{U} accepts iff \mathcal{A} accepts
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Every solvable problem can be solved in software.

Operation of \mathcal{U}

- 1 encode initial configuration
 - ▶ tape on lhs of head
 - ▶ state
 - ▶ tape on rhs of head
- 2 use $c(\mathcal{A})$ to find a transition from the current configuration
- 3 modify the current configuration accordingly
- 4 accept if \mathcal{A} accepts
- 5 stop if \mathcal{A} stops
- 6 otherwise, continue with step 2

The Halting problem

Definition (halting problem)

For a TM $\mathcal{A} = (Q, \Sigma, \Gamma, q_0, \Delta, F)$ and a word $w \in \Sigma^*$, does \mathcal{A} halt (i.e. reach a stop configuration) with input w ?

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- 1 \mathcal{U} (almost) does what $\mathcal{H}1$ needs to do.
- 2 Difficult: $\mathcal{H}2$ needs to detect that that \mathcal{A} does not terminate.
 - ▶ infinite tape \rightsquigarrow infinite number possible configurations
 - ▶ recognising repeated configurations not sufficient.

Undecidability of the halting problem

Assumption: there is a TM \mathcal{H}_2 which, given $c(\mathcal{A})$ and $c(w)$ as input

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- 3 afterwards \mathcal{S} operates like \mathcal{H}_2

Computation of \mathcal{S} with input $c(\mathcal{S})$

Reminder: \mathcal{S} accepts $c(\mathcal{A})$ iff \mathcal{A} does **not** accept $c(\mathcal{A})$.

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Theorem (Turing 1936)

The halting problem is undecidable.

Theorem (Decision problems for Turing Machines)

The word problem, the emptiness problem, and the equivalence problem are undecidable.

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Proof.

If any of these problems were decidable, one could easily derive a decision procedure for the halting problem. □

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Theorem (closure under $\bar{}$)

*The class of languages accepted by Turing Machines is **not closed** under complement.*

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*The class of languages accepted by TMs is **closed** under $\cup, \cdot, *, \cap$.*

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Proof.

Analogous to Type-1-grammars / LBAs.

Diagonalisation

Challenge of the proof:
show for all possible (infinitely many) TMs that none of them can
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TM	input	$c(A)$	$c(B)$	$c(C)$	$c(D)$	$c(E)$	\dots
A		\times					
B			\times				
C				\times			
D					\times		
E						\times	
\dots							\dots

Further diagonalisation arguments

Theorem (Cantor diagonalisation, 1891)

The set of real numbers is uncountable.

Theorem (Epimenides paradox, 6th century BC)

Epimenides [the Cretan] says: “[All] Cretans are always liars.”

Theorem (Russell's paradox, 1903)

$R := \{T \mid T \notin T\}$ Does $R \in R$ hold?

Theorem (Gödel's incompleteness theorem, 1931)

Construction of a sentence in 2nd order predicate logic which states that itself cannot be proved.

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Does it matter in practice?

It does not only affect halting

Halting is a fundamental property.

If halting cannot be decided, what can be?

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Theorem (Rice, 1953)

Every non-trivial semantic property of TMs is undecidable.

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non-trivial satisfied by some TMs, not satisfied by others

semantic referring to the accepted language

Undecidability of semantic properties

Example (Property E : TM accepts the set of prime numbers P)

If E is decidable, then so is the halting problem for \mathcal{A} and an input $w_{\mathcal{A}}$.

Approach: Turing Machine \mathcal{E} , input $w_{\mathcal{E}}$

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Check if \mathcal{E} accepts the set of prime numbers:

yes $\leadsto \mathcal{A}$ halts with input $w_{\mathcal{A}}$ no $\leadsto \mathcal{A}$ does not halt on input $w_{\mathcal{A}}$

It does not only affect Turing Machines

Church-Turing-thesis

Every effectively calculable function is a computable function.

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**No interesting property is decidable
for any powerful programming language!**

Undecidable problems in practice

- software development** Does the program match the specification?
- debugging** Does the program have a memory leak?
- malware** Does the program harm the system?
- education** Does the student's TM compute the same function as the teacher's TM?
- formal languages** Do two cf. grammars generate the same language?
- mathematics** Hilbert's tenth problem: find integer solutions for a polynomial with several variables
- logic** Satisfiability of formulas in first-order predicate logic

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Yes, it does matter!

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there will always be cases in which an incorrect answer or none at all is given.

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Turing Machines: summary

- ▶ Halting problem: does TM \mathcal{A} halt on input w ?
- ▶ Turing: no TM can decide the halting problem.
- ▶ Rice: no TM can decide any non-trivial semantic property of TMs.
- ▶ Church-Turing: this holds for every powerful machine model.
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Consequences:

- ☹ Computers cannot take all work away from computer scientists.
- 😊 Computers will never make computer scientists redundant.

Property overview

property	regular (Type 3)	context-free (Type 2)	context-sens. (Type 1)	unrestricted (Type 0)
closure				
$\cup, \cdot, *$	✓	✓	✓	✓
\cap	✓	✗	✓	✓
$_$	✓	✗	✓	✗
decidability				
word	✓	✓	✓	✗
emptiness	✓	✓	✗	✗
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End lecture 17

This is the End...

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Regular Languages and Finite Automata

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Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

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Lecture 2

Lecture 3

Lecture 4

Lecture 5

Lecture 6

Lecture 7

Lecture 8

Lecture 9

Lecture 10

Lecture 11

Lecture 12

Lecture 13

Lecture 14

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Lecture 18

Bonus Exercises

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Lecture 12

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Lecture 15

Lecture 16

Lecture 17

Lecture 18

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Goals for Lecture 1

- ▶ Getting acquainted
- ▶ Clarifying practical issues
- ▶ Course outline and motivation
 - ▶ Formal languages
 - ▶ Language classes
 - ▶ Grammars
 - ▶ Automata
 - ▶ Questions
 - ▶ Applications

- ▶ Lecture times (usually)
 - ▶ Tuesday 12:30-14:00
 - ▶ Wednesday 14:00-15:45
 - ▶ Plus irregular extra times (usually Monday)
 - ▶ No lectures:
 - ▶ KW39 (September 24.-28.)
 - ▶ September 26th, October 4th, October 25th, October 27th
 - ▶ KW44-KW45 (29.10.-11.11.)
- ▶ Written exam
 - ▶ Calendar week 47 (19.11.–23.11.)

Summary Lecture 1

- ▶ Clarifying practical issues
 - ▶ You need running `flex`, `bison`, C compiler, editor!
- ▶ Course outline and motivation
 - ▶ Formal languages
 - ▶ Language classes
 - ▶ Grammars
 - ▶ Automata
 - ▶ Questions
 - ▶ Applications

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 2

- ▶ Review of last lecture
- ▶ Formal basics of formal languages
- ▶ Operations on words
- ▶ Operations on languages
- ▶ Introduction to regular expressions
 - ▶ Examples
 - ▶ Formal definition

- ▶ Introduction
 - ▶ Language classes
 - ▶ Grammars
 - ▶ Automata
 - ▶ Applications

Summary

- ▶ Formal languages
 - ▶ Alphabets
 - ▶ Words
 - ▶ Languages
 - ▶ Examples of languages
- ▶ Operations on words and languages
 - ▶ Concatenation
 - ▶ Power
 - ▶ Kleene star
- ▶ Introduction to regular expressions

Summary

- ▶ Formal languages
 - ▶ Alphabets
 - ▶ Words
 - ▶ Languages
 - ▶ Examples of languages
- ▶ Operations on words and languages
 - ▶ Concatenation
 - ▶ Power
 - ▶ Kleene star
- ▶ Introduction to regular expressions

Remark: Formal languages are *sets*, and hence we can also apply set operations like \cup (union), \cap (intersection), $\bar{}$ (complement) to them!

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 3

- ▶ Review of last lecture
- ▶ Understanding regular expressions
- ▶ Regular expression algebra
 - ▶ Equivalences on regular expressions
 - ▶ Simplifying REs

Review (1)

- ▶ Formal languages
 - ▶ Finite **alphabet** Σ of symbols/letters
 - ▶ **Words** are finite sequences of letters from Σ
 - ▶ **Languages** are (finite or infinite) sets of words
- ▶ Words - properties and operations
 - ▶ $|w|, |w|_a, w[k]$
 - ▶ $w_1 \cdot w_2, w^n$

Review (2)

- ▶ Interesting languages
 - ▶ Binary representations of natural numbers
 - ▶ Binary representations of prime numbers
 - ▶ C functions (over strings)
 - ▶ C functions with input/output pairs
- ▶ Operations on Languages
 - ▶ Product $L_1 \cdot L_2$: Concatenation of one word from each language
 - ▶ Power L^n : Concatenation of n words from L
 - ▶ Kleene Star: L^* : Concat any number of words from L

Review (2)

- ▶ Interesting languages
 - ▶ Binary representations of natural numbers
 - ▶ Binary representations of prime numbers
 - ▶ C functions (over strings)
 - ▶ C functions with input/output pairs
- ▶ Operations on Languages
 - ▶ Product $L_1 \cdot L_2$: Concatenation of one word from each language
 - ▶ Power L^n : Concatenation of n words from L
 - ▶ Kleene Star: L^* : Concat any number of words from L

Remark: Formal languages are *sets*, and hence we can also apply set operations like \cup (union), \cap (intersection), $\bar{}$ (complement) to them!

- ▶ Regular expressions R_Σ
 - ▶ Base cases:
 - ▶ $L(\emptyset) = \{\}$
 - ▶ $L(\epsilon) = \{\epsilon\}$
 - ▶ $L(a) = \{a\}$ for each $a \in \Sigma$
 - ▶ Complex cases:
 - ▶ $L(r_1 + r_2) = L(r_1) \cup L(r_2)$
 - ▶ $L(r_1 \cdot r_2) = L(r_1 r_2) = L(r_1) \cdot L(r_2)$
 - ▶ $L(r^*) = L(r)^*$
 - ▶ $L((r)) = L(r)$ (brackets are used to group expressions)

- ▶ Regular expression algebra
 - ▶ REs are equivalent if they describe the same language
 - ▶ REs can be simplified applying equivalences
 - ▶ Recursive definitions: Lemmas of Arden/Salomaa

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 4

- ▶ Review of last lecture
- ▶ Warmup exercise
- ▶ Finite Automata
 - ▶ Graphical representation
 - ▶ Formal definition
 - ▶ Language recognized by an automata
 - ▶ Tabular representation
 - ▶ Exercises

▶ Regular expressions R_Σ

▶ Base cases:

▶ $L(\emptyset) = \{\}$

▶ $L(\epsilon) = \{\epsilon\}$

▶ $L(a) = \{a\}$ for each $a \in \Sigma$

▶ Complex cases:

▶ $L(r_1 + r_2) = L(r_1) \cup L(r_2)$

▶ $L(r_1 \cdot r_2) = L(r_1 r_2) = L(r_1) \cdot L(r_2)$

▶ $L(r^*) = L(r)^*$

▶ $L((r)) = L(r)$ (brackets are used to group expressions)

Review (2)

- ▶ Regular expression algebra
 - ▶ REs are equivalent if they describe the same language
 - ▶ REs can be simplified applying equivalences
 - ▶ 17 equivalences
 - ▶ Commutativity of $+$, associativity of $+$, \cdot , distributivity (!)
 - ▶ Arden/Salomaa
 - ▶ ...

Warmup Exercise

- ▶ Assume $\Sigma = \{a, b\}$
 - ▶ Find a regular expression for the language L_1 of all words over Σ with at least 3 characters and where the third character is a a .
 - ▶ Describe L_1 formally (i.e. as a set)
 - ▶ Find a regular expression for the language L_2 of all words over Σ with at least 3 characters and where the third character is the same as the third-last character
 - ▶ Describe L_2 formally.

Warmup Exercise

- ▶ Assume $\Sigma = \{a, b\}$
 - ▶ Find a regular expression for the language L_1 of all words over Σ with at least 3 characters and where the third character is a a .
 - ▶ Describe L_1 formally (i.e. as a set)
 - ▶ Find a regular expression for the language L_2 of all words over Σ with at least 3 characters and where the third character is the same as the third-last character
 - ▶ Describe L_2 formally.

- ▶ REs revisited (with exercise)
- ▶ Finite Automata
 - ▶ Graphical representation
 - ▶ Formal definition
 - ▶ Language recognized by an automata
 - ▶ Tabular representation
 - ▶ Exercises

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 5

- ▶ Review of last lecture
- ▶ Introduction to Nondeterministic Finite Automata
 - ▶ Definitions
 - ▶ Exercises
- ▶ Equivalency of deterministic and nondeterministic finite automata
 - ▶ Converting NFAs to DFAs
 - ▶ Exercises

- ▶ Finite Automata
 - ▶ Graphical representation
 - ▶ Formal definition
 - ▶ Q : Set of states
 - ▶ Σ : Alphabet
 - ▶ δ : Transition function
 - ▶ q_0 : Initial state
 - ▶ F : Final (accepting) states
 - ▶ Language recognized by an automata
 - ▶ $w \in L(A)$ if $\delta'(w) \in F$ or there exist an accepting run of A for w
 - ▶ Tabular representation
 - ▶ Exercises

- ▶ Finite Automata
 - ▶ Graphical representation
 - ▶ Formal definition
 - ▶ Q : Set of states
 - ▶ Σ : Alphabet
 - ▶ δ : Transition function
 - ▶ q_0 : Initial state
 - ▶ F : Final (accepting) states
 - ▶ Language recognized by an automata
 - ▶ $w \in L(A)$ if $\delta'(w) \in F$ or there exist an accepting run of A for w
 - ▶ Tabular representation
 - ▶ Exercises

- ▶ Nondeterministic Finite Automata
 - ▶ Q : Set of states
 - ▶ Σ : Alphabet
 - ▶ Δ : Transition *relation* (with ϵ -transitions!)
 - ▶ q_0 : Initial state
 - ▶ F : Final (accepting) states
- ▶ NFAs and DFAs accept the same class of languages!
- ▶ Converting NFA to DFA
 - ▶ States of $\text{det}(\mathcal{A})$ are elements of 2^Q
 - ▶ ...

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 6

- ▶ Review of last lecture
- ▶ Equivalency of regular expressions and NFAs
 - ▶ Construction of an NFA from a regular expression
 - ▶ Extraction of an RE from a DFA

- ▶ Nondeterministic Finite Automata
 - ▶ Q : Set of states
 - ▶ Σ : Alphabet
 - ▶ Δ : Transition *relation* (with ε -transitions!)
 - ▶ q_0 : Initial state
 - ▶ F : Final (accepting) states
- ▶ NFAs and DFAs accept the same class of languages!
- ▶ Converting NFA to DFA
 - ▶ States of $\det(\mathcal{A})$ are elements of 2^Q
 - ▶ ...

- ▶ Equivalency of regular expressions and NFAs
 - ▶ Construction of an NFA from a regular expression
 - ▶ Base cases each result in a trivial 2-state automaton
 - ▶ Compose automata for composite regular expressions
 - ▶ *Glue* automata together with ϵ -Transitions
 - ▶ Extraction of an RE for a DFA
 - ▶ Determine system of equations
 - ▶ For each state add one alternative for each transition
 - ▶ For accepting states add ϵ .
 - ▶ Solve the system of equations, handling loops with Arden's lemma

- ▶ What was the best part of todays lecture?
- ▶ What part of todays lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 7

- ▶ Review of last lecture
- ▶ Minimizing DFAs
 - ▶ ... and a first application
- ▶ Exercises

Review

- ▶ Central result: REs, NFAs and DFAs describe the same class of languages (namely **regular languages**)
 - ▶ Proof via constructive algorithms
 - ▶ NFA to DFA already done
 - ▶ RE to NFA and DFA to RE new
- ▶ Construction of an NFA from a regular expression
 - ▶ Base cases each result in a trivial 2-state automaton
 - ▶ Compose automata for composite regular expressions
 - ▶ *Glue* automata together with ϵ -Transitions
- ▶ Extraction of an RE for a DFA
 - ▶ Determine system of equations describing language accepted at each state
 - ▶ For each state add one alternative for each transition
 - ▶ For accepting states add ϵ .
 - ▶ Solve the system of equations, handling loops with Arden's lemma

Homework assignment

- ▶ Get access to an operational Linux/UNIX environment
 - ▶ You can install VirtualBox (<https://www.virtualbox.org>) and then install e.g. Ubuntu (<http://www.ubuntu.com/>) on a virtual machine
 - ▶ For Windows, you can install the **complete** UNIX emulation package Cygwin from <http://cygwin.com>
 - ▶ For MacOS, you can install `fink` (<http://fink.sourceforge.net/>) or MacPorts (<https://www.macports.org/>) and the necessary tools
- ▶ You will need at least `flex`, `bison`, `gcc`, `grep`, `sed`, `AWK`, `make`, and a good text editor of your choice

- ▶ Minimizing DFAs
 - ▶ Identify and merge equivalent states
 - ▶ Result is unique (up to names of states)
 - ▶ Equivalency of REs can be decided by comparison of corresponding minimal DFAs
- ▶ Homework: Get ready for `flexing...`

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 8

- ▶ Review of last lecture
- ▶ Beyond regular languages: The Pumping Lemma
 - ▶ Motivation/Lemma
 - ▶ Application of the lemma
 - ▶ Implications
- ▶ Properties of regular languages
 - ▶ Which operations leave regular languages regular?

- ▶ Minimizing DFAs
 - ▶ Identify and merge equivalent states
 - ▶ Result is unique (up to names of states)
 - ▶ Equivalency of REs can be decided by comparison of corresponding minimal DFAs

Reminder: Homework assignment

- ▶ Install an operational UNIX/Linux environment on your computer
 - ▶ You can install VirtualBox (<https://www.virtualbox.org>) and then install e.g. Ubuntu (<http://www.ubuntu.com/>) on a virtual machine
 - ▶ For Windows, you can install the **complete** UNIX emulation package Cygwin from <http://cygwin.com>
 - ▶ For MacOS, you can install `fink` (<http://fink.sourceforge.net/>) or MacPorts (<https://www.macports.org/>) and the necessary tools
- ▶ You will need at least `flex`, `bison`, `gcc`, `grep`, `sed`, `AWK`, `make`, and a good text editor of your choice

- ▶ Beyond regular languages: The Pumping Lemma
 - ▶ Motivation/Lemma
 - ▶ Application of the lemma ($a^n b^n, a^n b^m, n < m$)
 - ▶ Implications (Nested structures are not regular)
- ▶ Properties of regular languages
 - ▶ Closure properties (union, intersection, ...)
 - ▶ Proof per construction of suitable NFA (union, concatenation, Kleene Star)
 - ▶ Proof per construction of product automaton (intersection)

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 9

- ▶ Completing the theory of regular languages
 - ▶ Complement
 - ▶ Finite languages
 - ▶ Decision problems: Emptiness, word problem, . . .
 - ▶ Decision problems: Equivalence, finiteness, . . .
 - ▶ Wrap-up

▶ The Pumping Lemma

▶ Motivation/Lemma

- ▶ For every regular language L there exists a k such that any word s with $|s| \geq k$ can be split into $s = uvw$ with $|uv| \leq k$ and $v \neq \varepsilon$ and $uv^h w \in L$ for all $h \in \mathbb{N}$
- ▶ Use in proofs by contradiction: Assume a language is regular, then derive contradiction
- ▶ Application of the lemma ($a^n b^n, a^n b^m, n < m$)
- ▶ Implications (Nested structures are not regular)

▶ Properties of regular languages

- ▶ The union of two regular languages is regular
- ▶ The intersection of two regular languages is regular (Solution: Product automaton!)
- ▶ The concatenation of two regular languages is regular
- ▶ The Kleene star of a regular language is regular
- ▶ (The complement of a regular language is regular)

- ▶ Completing the theory of regular languages
 - ▶ Complement
 - ▶ All finite languages are regular
 - ▶ Decidable for regular languages: Emptiness, word problem
 - ▶ Decidable for regular languages: Equivalence, finiteness
 - ▶ Wrap-up

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 10

- ▶ Review of last lecture
- ▶ Scanning in practice
 - ▶ Scanners in context
 - ▶ Practical regular expressions
 - ▶ Automatic scanner creation with `flex`
 - ▶ `Flex` exercise

- ▶ Completing the theory of regular languages
 - ▶ Complement
 - ▶ All finite languages are regular
 - ▶ Decidable for regular languages: Emptiness, word problem
 - ▶ Decidable for regular languages: Equivalence, finiteness
 - ▶ Use of properties to prove statements (e.g. non-regularity, e.g. reducing emptiness question to equivalency)
 - ▶ Wrap-up

Summary

- ▶ Scanners in context
- ▶ Practical regular expressions
 - ▶ Basic characters, escapes, ranges, escape with \
 - ▶ Richer operators, z.B. +, {4},?...
- ▶ Flex
 - ▶ Definition section
 - ▶ Rule section
 - ▶ User code section/`yyllex()`
- ▶ Exercise

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 11

- ▶ Review of last lecture
- ▶ Grammars and languages
 - ▶ Formal grammars
 - ▶ Derivations
 - ▶ Languages
- ▶ The Chomsky-Hierarchy

Review

- ▶ Scanners in context
- ▶ Practical regular expressions
 - ▶ Basic characters, escapes, ranges, escape with \
 - ▶ Richer operators, z.B. +, {4},?...
- ▶ Flex
 - ▶ Definition section
 - ▶ Rule section
 - ▶ User code section/`yylex()`
- ▶ `make`
- ▶ Exercise

- ▶ Grammars and languages
 - ▶ Formal grammars
 - ▶ Derivations
 - ▶ Languages
 - ▶ Grammars can describe non-regular languages
- ▶ The Chomsky-Hierarchy
 - ▶ Type 0: Unrestricted grammars
 - ▶ Type 1: Context-sensitive/monotonic grammars
 - ▶ Type 2: Context-free grammars
 - ▶ Type 3: Right-linear/regular grammars

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 12

- ▶ Review of last lecture
- ▶ Right-linear grammars
 - ▶ Equivalence with finite automata
- ▶ Context-free grammars
 - ▶ Equivalency for grammars
 - ▶ Reducing grammars
 - ▶ Eliminating ϵ -Productions
 - ▶ Chomsky-hierarchy revisited

- ▶ Grammars and languages
 - ▶ Formal grammars (N, Σ, P, S)
 - ▶ Derivations
 - ▶ Languages $L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$
 - ▶ Grammars can describe non-regular languages
- ▶ The Chomsky-Hierarchy
 - ▶ Type 0: Unrestricted grammars
 - ▶ Type 1: Context-sensitive/monotonic grammars (non-shortening rules)
 - ▶ Type 2: Context-free grammars (Only one non-terminal on LHS)
 - ▶ Type 3: Right-linear/regular grammars (limited RHS)

Summary

- ▶ Right-linear grammars: $N \rightarrow aB, a \in \Sigma \cup \{\epsilon\}$
 - ▶ Convert DFA to right-linear grammar
 - ▶ Convert right-linear grammar to NFA
- ▶ Context-free grammars
 - ▶ Chomsky Normal Form
 - ▶ Towards CNF
 - ▶ Remove non-terminating symbols and corresponding rules
 - ▶ Remove non-reachable symbols and corresponding rules
 - ▶ Inline ϵ -rules

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Goals for Lecture 13

- ▶ Review of last lecture
- ▶ Completing Chomsky Normal Form transformation
 - ▶ Eliminating chain rules
 - ▶ Bring right hand sides into normalform
- ▶ Solving the word problem for context-free grammars
 - ▶ Derivations in CNF
 - ▶ Parsing with Dynamic Programming: Cocke-Younger-Kasami

- ▶ Right-linear grammars: $N \rightarrow aB, a \in \Sigma \cup \{\epsilon\}$
 - ▶ Convert DFA to right-linear grammar ($N \sim Q, \delta \sim P$)
 - ▶ Convert right-linear grammar to NFA
- ▶ Context-free grammars
 - ▶ Chomsky Normal Form
 - ▶ Towards CNF
 - ▶ Remove non-terminating symbols and corresponding rules
 - ▶ Remove non-reachable symbols and corresponding rules
 - ▶ Inline ϵ -rules
 - ▶ ...

- ▶ Chomsky Normal Form transformation
 - ▶ Inline ε -rules
 - ▶ Inline chain rules
 - ▶ Reduce grammar
 - ▶ Introduce NTS names for terminals
 - ▶ Break long RHS via introduction of definitions/intermediate rules
- ▶ Solving the word problem for context-free grammars
 - ▶ Derivations in CNF
 - ▶ Cocke-Younger-Kasami
 - ▶ Systematically consider all decompositions of w
 - ▶ Tabulate all NTS for given subwords bottom-up
 - ▶ Example of *dynamic programming*

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Regular Languages and Finite Automata

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Parsers and Bison

Turing Machines and Languages of Type 1 and 0

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Bonus Exercises

Selected Solutions

Goals for Lecture 14

- ▶ Review of last lecture
- ▶ Pushdown automata
 - ▶ NFA+unlimited stack
- ▶ Equivalence of context-free grammars and PDAs
 - ▶ From grammar to PDA
 - ▶ From PDA to grammar

- ▶ Chomsky Normal Form transformation
 - ▶ Inline ε -rules
 - ▶ Inline chain rules
 - ▶ Reduce grammar
 - ▶ Introduce NTS names for terminals
 - ▶ Break long RHS via introduction of definitions/intermediate rules
- ▶ Solving the word problem for context-free grammars
 - ▶ Derivations in CNF
 - ▶ Cocke-Younger-Kasami
 - ▶ Systematically consider all decompositions of w
 - ▶ Tabulate all NTS for given subwords bottom-up
 - ▶ Complexity: $O(n^3)$ with $n = |w|$

Summary

- ▶ Pushdown automata
 - ▶ Transitions must read/can write unlimited stack
 - ▶ Transition can read characters from word (left-to-right only)
 - ▶ Acceptance: Empty stack, empty word
- ▶ Equivalence of context-free grammars and PDAs
 - ▶ From grammar to PDA
 - ▶ Simulate grammar rules on the stack
 - ▶ Highly non-deterministic
 - ▶ From PDA to grammar
 - ▶ Non-terminals represent state changes with removal of a symbols from stack
 - ▶ Complex encoding of transition relation into grammar rules

Feedback round

- ▶ What was the best part of today's lecture?
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Bonus Exercises

Selected Solutions

Goals for Lecture 15

- ▶ Review of last lecture
- ▶ Limits of context-free languages
 - ▶ Pumping Lemma II
- ▶ Closure properties of context-free languages
- ▶ Decision problems for context-free languages

- ▶ Pushdown automata
 - ▶ Transitions must read/can write unlimited stack
 - ▶ Transition can read characters from word (left-to-right only)
 - ▶ Acceptance: Empty stack, empty word
- ▶ Equivalence of context-free grammars and PDAs
 - ▶ From grammar to PDA
 - ▶ Simulate grammar rules on the stack
 - ▶ Highly non-deterministic
 - ▶ From PDA to grammar
 - ▶ Non-terminals represent state changes with removal of a symbols from stack
 - ▶ Complex encoding of transition relation into grammar rules

Summary

- ▶ Limits of context-free languages
 - ▶ Pumping Lemma II
- ▶ Closure properties of context-free languages
 - ▶ The class of context-free languages is closed under $\cup, \cdot, *$
 - ▶ The class of context-free languages is not closed under \cap , complement
- ▶ Decision problems for context-free languages
 - ▶ Decidable: Word problem, emptiness problem
 - ▶ Undecidable: Equivalence

Feedback round

- ▶ What was the best part of today's lecture?
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Bonus Exercises

Selected Solutions

Goals for Lecture 16

- ▶ Review of last lecture
- ▶ Introduction to parsing
- ▶ YACC/Bison
 - ▶ Automatic parser generation
 - ▶ Example: desk calculator
- ▶ LALR(1) shift/reduce parsing
- ▶ Parse trees and abstract syntax trees

- ▶ Limits of context-free languages
 - ▶ Pumping Lemma II
 - ▶ Sufficiently long words require correspondingly long derivations
 - ▶ Sufficiently long derivations will contain at least one duplicate NTS A
 - ▶ The partial derivation from A to uAv can be repeated
- ▶ Closure properties of context-free languages
 - ▶ The class of context-free languages is closed under $\cup, \cdot, *$
 - ▶ The class of context-free languages is not closed under \cap , complement
- ▶ Decision problems for context-free languages
 - ▶ Decidable: Word problem, emptiness problem
 - ▶ Undecidable: Equivalence

Summary

- ▶ Introduction to parsing
 - ▶ Recognize words (programs) in $L(G)$
 - ▶ Understand structure of words
- ▶ YACC/Bison
 - ▶ Automatic parser generation
 - ▶ Core: Syntax rules with actions (C code)
 - ▶ Whenever input can be *reduced*, action is executed
 - ▶ Often: Manipulation of *semantic values*
 - ▶ Example: desk calculator
- ▶ LALR(1) shift/reduce parsing
 - ▶ **Shift** input to stack
 - ▶ **Reduce** RHS to LHS
 - ▶ Use lookahead to reduce ambiguity
- ▶ Parse trees and abstract syntax trees
 - ▶ Parse trees: Represents not just word but derivation
 - ▶ AST: “parse tree without syntactic details

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
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Bonus Exercises

Selected Solutions

Goals for Lecture 17

- ▶ Review of last lecture
- ▶ Beyond context-free languages: Turing Machines
 - ▶ Idea
 - ▶ Definition
 - ▶ Examples
 - ▶ Variants
- ▶ Context-sensitive languages
 - ▶ Linear bounded Turing Machine (LBA)
 - ▶ Closure properties
 - ▶ Decision problems
- ▶ Type-0 languages
 - ▶ The Universal Turing Machine
 - ▶ The Halting Problem
- ▶ Thesis of Church-Turing
 - ▶ ... and practical applications

Review

- ▶ Introduction to parsing
 - ▶ Recognize words (programs) in $L(G)$
 - ▶ Understand structure of words
- ▶ YACC/Bison
 - ▶ Automatic parser generation
 - ▶ Core: Syntax rules with actions (C code)
 - ▶ Whenever input can be *reduced*, action is executed
 - ▶ Often: Manipulation of *semantic values*
 - ▶ Example: desk calculator
- ▶ LALR(1) shift/reduce parsing
 - ▶ Shift input to stack
 - ▶ Reduce RHS to LHS
 - ▶ Use lookahead to reduce ambiguity
- ▶ Parse trees and abstract syntax trees
 - ▶ Parse trees: Represents not just word but derivation
 - ▶ AST: “parse tree without syntactic details

Summary

- ▶ Turing Machines
 - ▶ Unbounded tape memory
 - ▶ Finite automaton controls head movement, writing based on current state and input at current tape location
 - ▶ Configurations (general, stopping and accepting)
 - ▶ TM for $a^n b^n c^n$
- ▶ Context-sensitive languages
 - ▶ Linear bounded Turing Machine (LBA)
 - ▶ Closure properties: Closed under $\cup, \cdot, *, \cap, \bar{}$
 - ▶ Decision problems: Word yes, emptiness, equivalence no
- ▶ Type-0 languages
 - ▶ The Universal Turing Machine
 - ▶ The Halting Problem
- ▶ Thesis of Church-Turing: All “strong” computing models are equivalent
 - ▶ ... and so may be humans (!)

Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

Test Exam

Summary

- ▶ Review of last lecture
- ▶ Test exam
- ▶ Solutions

Final feedback round

- ▶ What was the best part of the **course**?
- ▶ What part of the course that has the most potential for improvement?
 - ▶ Optional: how would you improve it?

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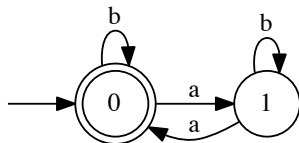
Bonus Exercises

Selected Solutions

Bonus Exercises

Assume $\Sigma = \{a, b\}$. Consider the automaton A .

- ▶ Give a formal description of A .
- ▶ Which language $L(A)$ is accepted by A ? Give a formal description.
- ▶ Where does $L(A)$ reside in the Chomsky-hierarchy?
- ▶ Manually find a regular expression R with $L(R) = L(A)$.
- ▶ Generate a system of equations for A and generate a regular expression R_S for $L(A)$ by solving this. Compare your result the the previous one.
- ▶ Convert A into a regular grammar G with $L(A) = L(G)$.



- ▶ Can $L(A)$ be pumped? If yes, provide an example of a pumpable word.
- ▶ Generalize A to recognize $L_3 = \{w \in \Sigma^* \mid |w|_a \text{ modulo } 3 = 0\}$
- ▶ Form the product automaton P to find $L(A) \cap L_3$.
- ▶ Minimize P .
- ▶ Systematically construct a NFA D_A for R_S .
- ▶ Convert D_R to a DFA.

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Selected Solutions

Equivalence of regular expressions

Solution to Exercise: Algebra on regular expressions (1)

► Claim: $r^* \doteq \varepsilon + r^*$

$$\varepsilon + r^* \doteq \varepsilon + \varepsilon + r^*r \quad (13)$$

Proof: $\doteq \varepsilon + r^*r \quad (9)$

$$\doteq r^* \quad (13)$$

Simplification of regular expressions

Solution to Exercise: Algebra on regular expressions (2)

$$\begin{aligned}r &= 0(\varepsilon + 0 + 1)^* + (\varepsilon + 1)(1 + 0)^* + \varepsilon \\ &\stackrel{14,1}{\doteq} 0(0 + 1)^* + (\varepsilon + 1)(0 + 1)^* + \varepsilon \\ &\stackrel{7}{\doteq} 0(0 + 1)^* + \varepsilon(0 + 1)^* + 1(0 + 1)^* + \varepsilon \\ &\stackrel{5}{\doteq} 0(0 + 1)^* + (0 + 1)^* + 1(0 + 1)^* + \varepsilon \\ &\stackrel{1,7}{\doteq} \varepsilon + (0 + 1)(0 + 1)^* + (0 + 1)^* \\ &\stackrel{16}{\doteq} \varepsilon + (0 + 1)^*(0 + 1) + (0 + 1)^* \\ &\stackrel{13}{\doteq} (0 + 1)^* + (0 + 1)^* \\ &\stackrel{9}{\doteq} (0 + 1)^*.\end{aligned}$$

Application of Arto's lemma

Solution to Exercise: Algebra on regular expressions (3)

▶ Show that $u = 10(10)^* \doteq 1(01)^*0$

▶ Idea: u is of the form ts^* with:

▶ $t = 10$

▶ $s = 10$

▶ This suggest Arto's Lemma. To apply the lemma, we must show that $r = 1(01)^*0 \doteq rs + t$

$$\begin{aligned}rs + t &= 1(01)^*010 + 10 \\ &\doteq 1((01)^*010 + 0) \quad \text{(factor out 1)}\end{aligned}$$

▶ So: $\doteq 1((01)^*01 + \varepsilon)0 \quad \text{(factor out 0)}$

$$\doteq 1(01)^*0 \quad (14)$$

$$= r$$

▶ Since $L(s) = \{10\}$ (and hence $\varepsilon \notin L(s)$), we can apply Arto and rewrite $r \doteq ts^* \doteq 10(10)^*$.

Transformation into DFA (1)

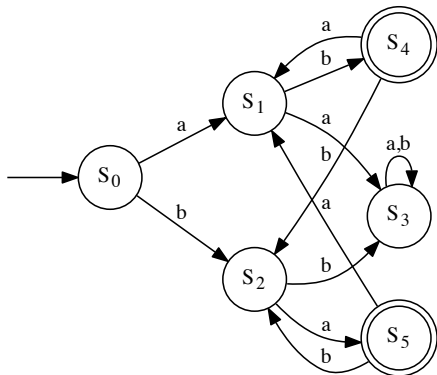
- ▶ Incremental computation of \hat{Q} and $\hat{\delta}$:
 - ▶ Initial state $S_0 = ec(q_0) = \{q_0, q_1, q_2\}$
 - ▶ $\hat{\delta}(S_0, a) = \delta^*(q_0, a) \cup \delta^*(q_1, a) \cup \delta^*(q_2, a) = \{\} \cup \{\} \cup \{q_4\} = \{q_4\} = S_1$
 - ▶ $\hat{\delta}(S_0, b) = \{q_3\} = S_2$
 - ▶ $\hat{\delta}(S_1, a) = \{\} = S_3$
 - ▶ $\hat{\delta}(S_1, b) = ec(q_6) = \{q_6, q_7, q_0, q_1, q_2\} = S_4$
 - ▶ $\hat{\delta}(S_2, a) = \{q_5, q_7, q_0, q_1, q_2\} = S_5$
 - ▶ $\hat{\delta}(S_2, b) = \{\} = S_3$
 - ▶ $\hat{\delta}(S_3, a) = \{\} = S_3$
 - ▶ $\hat{\delta}(S_3, b) = \{\} = S_3$
 - ▶ $\hat{\delta}(S_4, a) = \{q_4\} = S_1$
 - ▶ $\hat{\delta}(S_4, b) = \{q_3\} = S_2$
 - ▶ $\hat{\delta}(S_5, a) = \{q_4\} = S_1$
 - ▶ $\hat{\delta}(S_5, b) = \{q_3\} = S_2$
- ▶ $\hat{F} = \{S_4, S_5\}$ (since $q_7 \in S_4, q_7 \in S_5$)

Transformation into DFA (2)

- ▶ $det(\mathcal{A}) = (\hat{Q}, \Sigma, \hat{\delta}, S_0, \hat{F})$
 - ▶ $\hat{Q} = \{S_0, S_1, S_2, S_3, S_4, S_5\}$
 - ▶ $\hat{F} = \{S_4, S_5\}$
 - ▶ $\hat{\delta}$ given by the table below

$\hat{\delta}$	a	b
$\rightarrow S_0$	S_1	S_2
S_1	S_3	S_4
S_2	S_5	S_3
S_3	S_3	S_3
$*S_4$	S_1	S_2
$*S_5$	S_1	S_2

- ▶ Regexp:
 $L(\mathcal{A}) = L((ab + ba)(ab + ba)^*)$

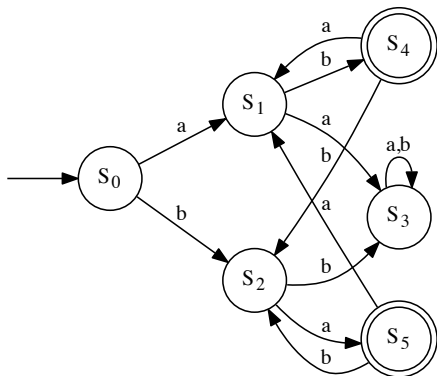


Transformation into DFA (2)

- ▶ $det(\mathcal{A}) = (\hat{Q}, \Sigma, \hat{\delta}, S_0, \hat{F})$
 - ▶ $\hat{Q} = \{S_0, S_1, S_2, S_3, S_4, S_5\}$
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$*S_5$	S_1	S_2

- ▶ Regexp:
 $L(\mathcal{A}) = L((ab + ba)(ab + ba)^*)$

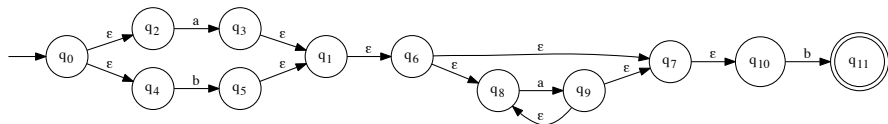


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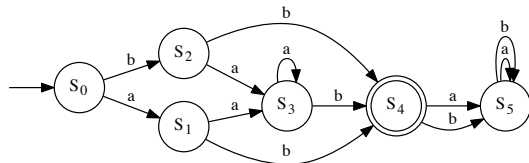
Transformation of RE into NFA

Systematically construct an NFA accepting the same language as the regular expression $(a + b)a^*b$.

Solution:

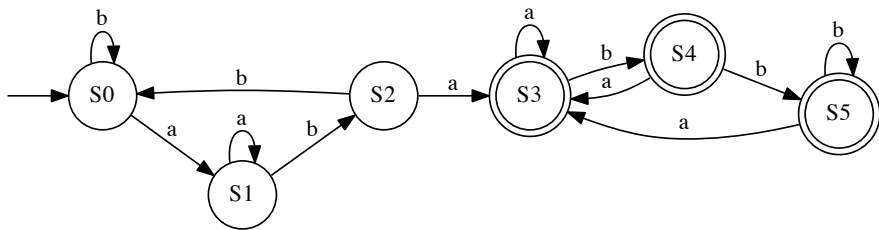


Corresponding DFA:



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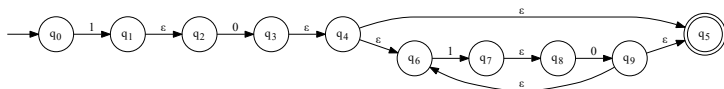
Solution: NFA to DFA “aba”



Show $10(10)^* \doteq 1(01)^*0$ via minimal DFAs (1)

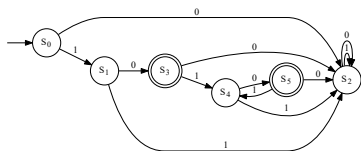
► Step 1: NFA for $10(10)^*$:

	epsilon	0	1
-> q0	{}	{}	{q1}
q1	{q2}	{}	{}
q2	{}	{q3}	{}
q3	{q4}	{}	{}
q4	{q5, q6}	{}	{}
* q5	{}	{}	{}
q6	{}	{}	{q7}
q7	{q8}	{}	{}
q8	{}	{q9}	{}
q9	{q5, q6}	{}	{}



Show $10(10)^* \doteq 1(01)^*0$ via minimal DFAs (2)

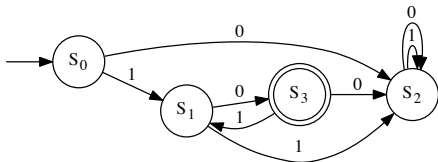
- Step 2: DFA \mathcal{A} for $10(10)^*$:



- Step 3: Minimizing of \mathcal{A}

	S_0	S_1	S_2	S_3	S_4	S_5
S_0	o	x	x	x	x	x
S_1	x	o	x	x	o	x
S_2	x	x	o	x	x	x
S_3	x	x	x	o	x	o
S_4	x	o	x	x	o	x
S_5	x	x	x	o	x	o

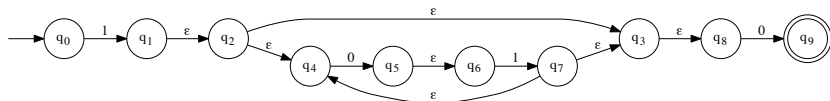
Result: (S_1, S_4) and (S_3, S_5) can be merged



Show $10(10)^* \doteq 1(01)^*0$ via minimal DFAs (3)

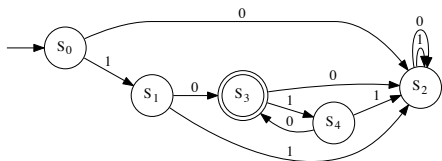
► Step 4: NFA zu $1(01)^*0$:

	epsilon	0	1
-> q0	{}	{}	{q1}
q1	{q2}	{}	{}
q2	{q3, q4}	{}	{}
q3	{q8}	{}	{}
q4	{}	{q5}	{}
q5	{q6}	{}	{}
q6	{}	{}	{q7}
q7	{q4, q3}	{}	{}
q8	{}	{}	{q9}
* q9	{}	{}	{}



Show $10(10)^* \doteq 1(01)^*0$ via minimal DFAs (4)

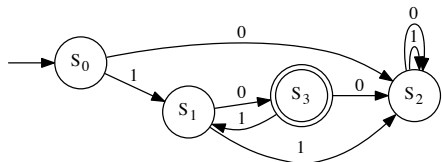
- Step 5: DFA \mathcal{B} for $1(01)^*0$



- Step 6: Minimization of \mathcal{B}

	S_0	S_1	S_2	S_3	S_4
S_0	0	X	X	X	X
S_1	X	0	X	X	0
S_2	X	X	0	X	X
S_3	X	X	X	0	X
S_4	X	0	X	X	0

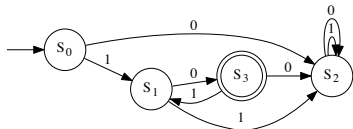
Result: (S_1, S_4) can be merged



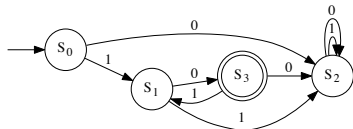
Show $10(10)^* \doteq 1(01)^*0$ via minimal DFAs (5)

- ▶ Step 7: Comparison of \mathcal{A}^- and \mathcal{B}^-

\mathcal{A}^-



\mathcal{B}^-



- ▶ Result: The two automata are identical, hence the two original regular expressions describe the same languages.

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Pumping lemma

Solution to $a^n b^m$ with $n < m$

- ▶ Proposition: $L = \{a^n b^m \mid n < m\}$ is not regular.
- ▶ Proof by contradiction. We assume L is regular
- ▶ Then: $\exists k \in \mathbb{N}$ with:
 - ▶ $\forall s \in L$ with $|s| \geq k : \exists u, v, w \in \Sigma^*$ such that
 - ▶ $s = uvw$
 - ▶ $|uv| \leq k$
 - ▶ $v \neq \varepsilon$
 - ▶ $uv^h w \in L$ for all $h \in \mathbb{N}$
- ▶ We consider the word $s = a^k b^{k+1} \in L$
 - ▶ Since $|uv| \leq k$: $u = a^i, v = a^j, w = a^l b^{k+1}$ and $j > 0, i + j + l = k$
 - ▶ Now consider $s' = uv^2 w$. According to the pumping lemma, $s' \in L$. But $s' = a^i a^j a^j a^l b^{k+1} = a^{i+j+l+j} b^{k+1} = a^{k+j} b^{k+1}$. Since $j \in \mathbb{N}, j > 0$: $k + j \not< k + 1$. Hence $s' \notin L$. This is a contradiction. Hence the assumption is wrong, and the original proposition is true. q.e.d.

Pumping lemma

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 - ▶ $uv^h w \in L$ for all $h \in \mathbb{N}$
- ▶ We consider the word $s = a^k b^{k+1} \in L$
 - ▶ Since $|uv| \leq k$: $u = a^i, v = a^j, w = a^l b^{k+1}$ and $j > 0, i + j + l = k$
 - ▶ Now consider $s' = uv^2 w$. According to the pumping lemma, $s' \in L$. But $s' = a^i a^j a^j a^l b^{k+1} = a^{i+j+l+j} b^{k+1} = a^{k+j} b^{k+1}$. Since $j \in \mathbb{N}, j > 0$: $k + j \not< k + 1$. Hence $s' \notin L$. This is a contradiction. Hence the assumption is wrong, and the original proposition is true. q.e.d.

Solution: Pumping lemma (Prime numbers)

- ▶ Proposition: $L = \{a^p \mid p \in \mathbb{P}\}$ is not regular (where \mathbb{P} is the set of all prime numbers)
- ▶ Proof: By contradiction, using the pumping lemma.
 - ▶ Assumption: L is regular. Then there exist a k such that all words in L with at least length k can be pumped.
- ▶ Consider the word $s = a^p$, where $p \in \mathbb{P}, p \geq k$
 - ▶ Then there are $u, v, w \in \Sigma^*$ with $uvw = s, |uv| \leq k, v \neq \varepsilon$, and $uv^h w \in L$ for all $h \in \mathbb{N}$.
 - ▶ We can write $u = a^i, v = a^j, w = a^l$ with $i + j + l = p$
 - ▶ So $s = a^i a^j a^l$ and $a^i a^{j \cdot h} a^l \in L$ for all $h \in \mathbb{N}$.
 - ▶ Consider $h = p + 1$. Then $a^i a^{j \cdot (p+1)} a^l \in L$
 - ▶ $a^i a^{j \cdot (p+1)} a^l = a^i a^{jp+j} a^l = a^i a^{jp} a^j a^l = a^i a^j a^l a^{jp} = a^p a^{jp} = a^{(j+1)p}$
 - ▶ But $(j+1)p \notin \mathbb{P}$, since $j+1 > 1$ and $p > 1$, and $(j+1)p$ thus has (at least) two non-trivial divisors.
 - ▶ Thus $a^{(j+1)p} \notin L$. This violates the pumping lemma and hence contradicts the assumption. Thus the assumption is wrong and the proposition holds. *q.e.d.*

Solution: Pumping lemma (Prime numbers)

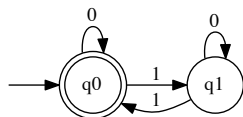
- ▶ Proposition: $L = \{a^p \mid p \in \mathbb{P}\}$ is not regular (where \mathbb{P} is the set of all prime numbers)
- ▶ Proof: By contradiction, using the pumping lemma.
 - ▶ Assumption: L is regular. Then there exist a k such that all words in L with at least length k can be pumped.
- ▶ Consider the word $s = a^p$, where $p \in \mathbb{P}, p \geq k$
 - ▶ Then there are $u, v, w \in \Sigma^*$ with $uvw = s, |uv| \leq k, v \neq \varepsilon$, and $uv^h w \in L$ for all $h \in \mathbb{N}$.
 - ▶ We can write $u = a^i, v = a^j, w = a^l$ with $i + j + l = p$
 - ▶ So $s = a^i a^j a^l$ and $a^i a^{j \cdot h} a^l \in L$ for all $h \in \mathbb{N}$.
 - ▶ Consider $h = p + 1$. Then $a^i a^{j \cdot (p+1)} a^l \in L$
 - ▶ $a^i a^{j \cdot (p+1)} a^l = a^i a^{jp+j} a^l = a^i a^{jp} a^j a^l = a^i a^j a^l a^{jp} = a^p a^{jp} = a^{(j+1)p}$
 - ▶ But $(j+1)p \notin \mathbb{P}$, since $j+1 > 1$ and $p > 1$, and $(j+1)p$ thus has (at least) two non-trivial divisors.
 - ▶ Thus $a^{(j+1)p} \notin L$. This violates the pumping lemma and hence contradicts the assumption. Thus the assumption is wrong and the proposition holds.

q.e.d.

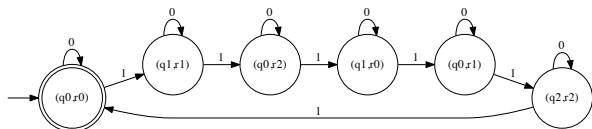
Solution: Product automaton for $L_1 \cap L_2$

Solution to Exercise: Product automaton

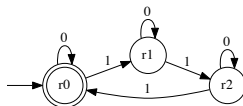
A_1 :



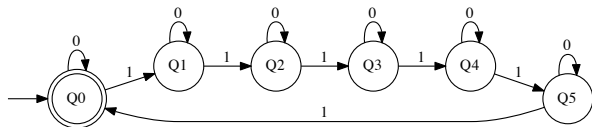
$A_1 \times A_2$ for $L_1 \cap L_2$:



A_2 :



Äquivalent Automaton for $L_1 \cap L_2$:



Solution: Transformation to Chomsky Normal Form (1)

Compute the Chomsky normal form of the following grammar:

$$G = (N, \Sigma, P, S)$$

▶ $N = \{S, A, B, C, D, E\}$

▶ $\Sigma = \{a, b\}$

$$S \rightarrow AB|SB|BDE$$

$$C \rightarrow SB$$

▶ $P :$ $A \rightarrow Aa$

$$D \rightarrow E$$

$$B \rightarrow bB|BaB|ab$$

$$E \rightarrow \varepsilon$$

Step 1: ε -Elimination

▶ Nullable NTS: $N = \{E, D\}$

$$S \rightarrow BD \quad (\text{from } S \rightarrow BDE, \beta_1 = BD, \beta_2 = \varepsilon)$$

▶ New rules: $S \rightarrow BE \quad (\text{from } S \rightarrow BDE, \beta_1 = B, \beta_2 = E)$

$$S \rightarrow B \quad (\text{from } S \rightarrow BD \text{ or } S \rightarrow BE, \beta_1 = B, \beta_2 = \varepsilon)$$

$$D \rightarrow \varepsilon \quad (\text{from } D \rightarrow E, \beta_1 = \varepsilon, \beta_2 = \varepsilon)$$

▶ Remove $E \rightarrow \varepsilon, D \rightarrow \varepsilon$

Solution: Transformation to Chomsky Normal Form (2)

Step 2: Elimination of Chain Rules.

- ▶ Current chain rules: $S \rightarrow B, D \rightarrow E$
- ▶ Eliminate $S \rightarrow B$:
 - ▶ $N(S) = \{B\}$
 - ▶ New rules: $S \rightarrow bB, S \rightarrow BaB, S \rightarrow ab$
- ▶ Eliminate $D \rightarrow E$
 - ▶ $N(D) = \{E\}$
 - ▶ E has no rule, therefore no new rules!
- ▶ Current state of P :

$S \rightarrow AB|SB|BDE|BD|BE|bB|BaB|ab$
 $A \rightarrow Aa$

$C \rightarrow SB$
 $B \rightarrow bB|BaB|ab$

Solution: Transformation to Chomsky Normal Form (3)

Step 3: Reducing the grammar

- ▶ Terminating symbols: $T = \{S, B, C\}$ (A, D, E do not terminate)

- ▶ Remove all rules that contain A, E, D . Remaining:

$$S \rightarrow SB|bB|BaB|ab \quad C \rightarrow SB$$

$$B \rightarrow bB|BaB|ab$$

- ▶ Reachable symbols: $R = \{S, B\}$ (C is not reachable)

- ▶ Remove all rules containing C . Remaining:

$$S \rightarrow SB|bB|BaB|ab$$

$$B \rightarrow bB|BaB|ab$$

Solution: Transformation to Chomsky Normal Form (4)

Step 4: Introduce new non-terminals for terminals

- ▶ New rules: $X_a \rightarrow a, X_b \rightarrow b$. Result:

$$\begin{array}{ll} S & \rightarrow SB|X_bB|BX_aB|X_aX_b & X_a & \rightarrow a \\ B & \rightarrow X_bB|BX_aB|X_aX_b & X_b & \rightarrow b \end{array}$$

Step 5: Introduce new non-terminals to break up long right hand sides:

- ▶ Problematic RHS: BX_aB (in two rules)
- ▶ New rule: $C_1 \rightarrow X_aB$. Result:

$$\begin{array}{ll} S & \rightarrow SB|X_bB|BC_1|X_aX_b & X_a & \rightarrow a \\ B & \rightarrow X_bB|BC_1|X_aX_b & X_b & \rightarrow b \\ C_1 & \rightarrow X_aB & & \end{array}$$

Solution: Transformation to Chomsky Normal Form (5)

Final grammar: $G' = (N', \Sigma, P', S)$ with

▶ $N' = \{S, B, C_1, X_a, X_b\}$

▶ $\Sigma = \{a, b\}$

▶ $P' :$

S	\rightarrow	$SB X_bB BC_1 X_aX_b$	X_a	\rightarrow	a
B	\rightarrow	$X_bB BC_1 X_aX_b$	X_b	\rightarrow	b
C_1	\rightarrow	X_aB			

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Solution: PDA to Grammar (1)

$$A = (Q, \Sigma, \Gamma, \Delta, 0, Z)$$

- ▶ $Q = \{0, 1\}$

- ▶ $\Sigma = \{a, b\}$

- ▶ $\Gamma = \{A, Z\}$

$$\Delta :$$

(1)	0	ϵ	Z	\rightarrow	ϵ	0
(2)	0	a	Z	\rightarrow	AZ	0
(3)	0	a	A	\rightarrow	AA	0
(4)	0	b	A	\rightarrow	ϵ	1
(5)	1	b	A	\rightarrow	ϵ	1
(6)	1	ϵ	Z	\rightarrow	ϵ	1

$$G = (N, \Sigma, P, S)$$

- ▶ $N = \{S, [0A0], [0A1], [1A0], [1A1], [0Z0], [0Z1], [1Z0], [1Z1]\}$

- ▶ Σ and S as given

- ▶ Start rules for P :

- ▶ $S \rightarrow [0Z0]$

- ▶ $S \rightarrow [0Z1]$

- ▶ From transitions:

- ▶ From (1): $[0Z0] \rightarrow \epsilon$

- ▶ From (4): $[0A1] \rightarrow b$

- ▶ From (5): $[1A1] \rightarrow b$

- ▶ From (6): $[1Z1] \rightarrow \epsilon$

- ▶ (2) and (3): Next page

Solution: PDA to Grammar (2)

$$\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, 0, Z)$$

▶ $Q = \{0, 1\}$

▶ $\Sigma = \{a, b\}$

▶ $\Gamma = \{A, Z\}$

$$\Delta :$$

(1)	0	ε	Z	\rightarrow	ε	0
(2)	0	a	Z	\rightarrow	AZ	0
(3)	0	a	A	\rightarrow	AA	0
(4)	0	b	A	\rightarrow	ε	1
(5)	1	b	A	\rightarrow	ε	1
(6)	1	ε	Z	\rightarrow	ε	11

Computing of P continued:

▶ From (2):

▶ $[0Z0] \rightarrow a[0A0][0Z0]$

▶ $[0Z0] \rightarrow a[0A1][1Z0]$

▶ $[0Z1] \rightarrow a[0A0][0Z1]$

▶ $[0Z1] \rightarrow a[0A1][1Z1]$

▶ From (3):

▶ $[0A0] \rightarrow a[0A0][0A0]$

▶ $[0A0] \rightarrow a[0A1][1A0]$

▶ $[0A1] \rightarrow a[0A0][0A1]$

▶ $[0A1] \rightarrow a[0A1][1A1]$

Solution: PDA to Grammar (3)

Full grammar $G = \{N, \Sigma, P, S\}$

▶ $N = \{S, [0A0], [0A1], [1A0], [1A1], [0Z0], [0Z1], [1Z0], [1Z1]\}$

▶ $\Sigma = \{a, b\}$

▶ P :

▶ $S \rightarrow [0Z0]$

▶ $S \rightarrow [0Z1]$

▶ $[0Z0] \rightarrow \varepsilon$

▶ $[0A1] \rightarrow b$

▶ $[1A1] \rightarrow b$

▶ $[1Z1] \rightarrow \varepsilon$

▶ $[0Z0] \rightarrow a[0A0][0Z0]$

▶ $[0Z0] \rightarrow a[0A1][1Z0]$

▶ $[0Z1] \rightarrow a[0A0][0Z1]$

▶ $[0Z1] \rightarrow a[0A1][1Z1]$

▶ $[0A0] \rightarrow a[0A0][0A0]$

▶ $[0A0] \rightarrow a[0A1][1A0]$

▶ $[0A1] \rightarrow a[0A0][0A1]$

▶ $[0A1] \rightarrow a[0A1][1A1]$

▶ **Terminating:** $T = \{[0Z0], [0A1], [1A1], [1Z1], S, [0Z1]\}$

▶ **Remaining rules:**

1 $S \rightarrow [0Z0]$

2 $S \rightarrow [0Z1]$

3 $[0Z0] \rightarrow \varepsilon$

4 $[0A1] \rightarrow b$

5 $[1A1] \rightarrow b$

6 $[1Z1] \rightarrow \varepsilon$

7 $[0Z1] \rightarrow a[0A1][1Z1]$

8 $[0A1] \rightarrow a[0A1][1A1]$

▶ **Reachable:**

$R = \{S, [0Z0], [0Z1], [0A1], [1Z1], [1A1]\}$

▶ No change!

Solution: PDA to Grammar (4)

► P :

1 $S \rightarrow [0Z0]$

2 $S \rightarrow [0Z1]$

3 $[0Z0] \rightarrow \varepsilon$

4 $[0A1] \rightarrow b$

5 $[1A1] \rightarrow b$

6 $[1Z1] \rightarrow \varepsilon$

7 $[0Z1] \rightarrow a[0A1][1Z1]$

8 $[0A1] \rightarrow a[0A1][1A1]$

► Derivation of ε :

► $S \Rightarrow_1 [0Z0] \Rightarrow_3 \varepsilon$

► Derivation of ab :

► $S \Rightarrow_2 [0Z1] \Rightarrow_7 a[0A1][1Z1] \Rightarrow_4$
 $ab[1Z1] \Rightarrow_6 ab$

► Derivation of $aabb$:

► $S \Rightarrow_2 [0Z1] \Rightarrow_7 a[0A1][1Z1] \Rightarrow_8$
 $aa[0A1][1A1][1Z1] \Rightarrow_4$
 $aab[1A1][1Z1] \Rightarrow_5 aabb[1Z1] \Rightarrow_6$
 $aabb$

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