

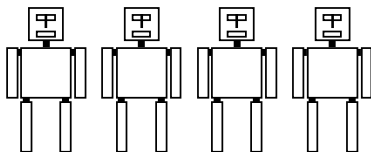
# Formal Languages and Automata

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with contributions from David Suendermann



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- ▶ Stephan Schulz
  - ▶ Dipl.-Inform., U. Kaiserslautern, 1995
  - ▶ Dr. rer. nat., TU München, 2000
  - ▶ Visiting professor, U. Miami, 2002
  - ▶ Visiting professor, U. West Indies, 2005
  - ▶ Lecturer (Hildesheim, Offenburg, ...) since 2009
  - ▶ Industry experience: Building Air Traffic Control systems
    - ▶ System engineer, 2005
    - ▶ Project manager, 2007
    - ▶ Product Manager, 2013
  - ▶ Professor, DHBW Stuttgart, 2014

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**Research: Logic & Automated Reasoning**

- ▶ Jan Hladik
  - ▶ Dipl.-Inform.: RWTH Aachen, 2001
  - ▶ Dr. rer. nat.: TU Dresden, 2007
  - ▶ Industry experience: SAP Research
    - ▶ Work in publicly funded research projects
    - ▶ Collaboration with SAP product groups
    - ▶ Supervision of Bachelor, Master, and PhD students
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**Research: Semantic Web, Semantic Technologies,  
Automated Reasoning**



## ▶ Scripts

- ▶ The most up-to-date version of this document as well as auxiliary material will be made available online at

`http://www.lehre.dhbw-stuttgart.de/~sschulz/fla2020.html`

and

`http://www.lehre.dhbw-stuttgart.de/~hladik/FLA`

## ▶ Books

- ▶ John E. Hopcroft, Rajeev Motwani, Jeffrey D. Ullman: [Introduction to Automata Theory, Languages, and Computation](#)
- ▶ Michael Sipser: [Introduction to the Theory of Computation](#)
- ▶ Dirk W. Hoffmann: [Theoretische Informatik](#)
- ▶ Ulrich Hedtstück: [Einführung in die theoretische Informatik](#)

# Computing Environment

- ▶ For practical exercises, you will need a Linux/UNIX environment. If you do not run one natively, there are several options:
  - ▶ You can install VirtualBox (<https://www.virtualbox.org>) and then install e.g. Ubuntu (<http://www.ubuntu.com/>) on a virtual machine
  - ▶ For Windows, you can install the **complete** UNIX emulation package Cygwin from <http://cygwin.com>
  - ▶ For MacOS, you can install MacPorts (<https://www.macports.org/>) or Homebrew (<https://brew.sh>) and the necessary tools
- ▶ You will need at least `flex`, `bison`, `gcc`, `make`, and a good text editor
- ▶ There are some example programs in Python on the course web page

# Outline of the Lecture

## Introduction

- Organisation
- Formal languages overview
- Formal language basics

## Regular Languages and Finite Automata

- Regular Expressions
- Finite Automata
- The Pumping Lemma
- Properties of Regular Languages

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# Formal language concepts

**Alphabet:** finite set  $\Sigma$  of symbols (characters)

▶  $\{a, b, c\}$

**Word:** finite sequence  $w$  of characters (string)

▶  $ab \neq ba$

**Language:** (possibly infinite) set  $L$  of words

▶  $\{ab, ba, abc\} = \{ba, abc, ab\}$

**Formal:**  $L$  defined precisely

▶ in contrast to **natural** languages, where there are borderline cases

# Some formal languages

## Example

- ▶ (the set of all) names in a phone directory
- ▶ (the set of all) phone numbers in a phone directory
- ▶ (the set of all) legal C identifiers
- ▶ (the set of all) legal C programs
- ▶ (the set of all) legal [HTML 4.01 Transitional](#) documents
- ▶ (the set of all) empty set
- ▶ (the set of all) ASCII strings
- ▶ (the set of all) Unicode strings

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**More?**

# Language classes

This course: four classes of different complexity and expressivity

- 1 **regular** languages: limited power, but easy to handle
  - ▶ “strings that start with a letter, followed by up to 7 letters or digits”
  - ▶ legal C identifiers
  - ▶ phone numbers



# Language classes

This course: four classes of different complexity and expressivity

- 1 regular** languages: limited power, but easy to handle
  - ▶ “strings that start with a letter, followed by up to 7 letters or digits”
  - ▶ legal C identifiers
  - ▶ phone numbers
- 2 context-free** languages: more expressive, but still feasible
  - ▶ “every `<token>` is matched by `</token>`”
  - ▶ **nested** dependencies
  - ▶ (most aspects of) legal C programs
  - ▶ many natural languages (English, German)

Jan says that we  
let  
the children  
help  
Hans  
paint  
the house

Jan sagt, dass wir  
die Kinder  
dem Hans  
das Haus  
anstreichen  
helfen  
ließen

## Language classes (cont')

- 3 **context-sensitive** languages: even more expressive, difficult to handle computationally
  - ▶ “every variable has to be declared before it is used”  
(arbitrary sequence, arbitrary amounts of code in between)
  - ▶ **cross-serial** dependencies
  - ▶ (remaining aspects of) legal C programs
  - ▶ most remaining natural languages

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```
Jan säit das mer  
d' chind  
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    aastriche
```

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- 4 **recursively enumerable** languages: most general (Chomsky) class; undecidable
  - ▶ all (valid) mathematical theorems (in first-order logic)
  - ▶ programs terminating on a particular input

# Automata

- ▶ Automata are **formal models** for computation
- ▶ Automata are characterised by
  - ▶ A set of **states**
  - ▶ Rules for the **transition** between states
  - ▶ A set of **letters** or **characters** (or, for control automata, **events**)
  - ▶ External **memory**

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  - ▶ External **memory**
- ▶ Two main applications for automata
  - ▶ *Control*: process sequences of events
  - ▶ *Languages* (focus of this course): process finite sequences (words) of characters

- ▶ Automata for language processing. . .
  - ▶ . . . process words
  - ▶ . . . potentially **accept** words



# Automata and Languages

- ▶ Automata for language processing. . .
  - ▶ . . . process words
  - ▶ . . . potentially **accept** words

For every language class discussed in this course, a machine model exists such that for every **language**  $L$  there is an **automaton**  $\mathcal{A}(L)$  that accepts exactly the words in  $L$ .

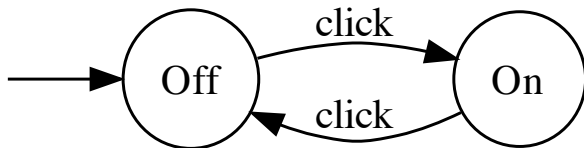
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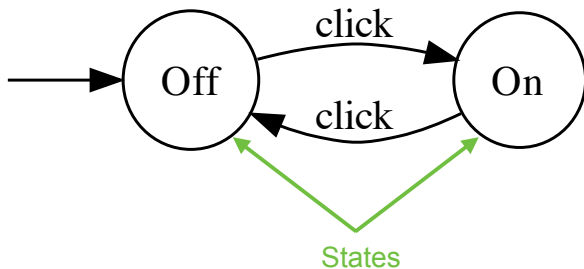
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regular	$\rightsquigarrow$	finite automaton/finite state machine
context-free	$\rightsquigarrow$	pushdown automaton
context-sensitive	$\rightsquigarrow$	linearly bounded Turing machine
recursively enumerable	$\rightsquigarrow$	(unbounded) Turing machine

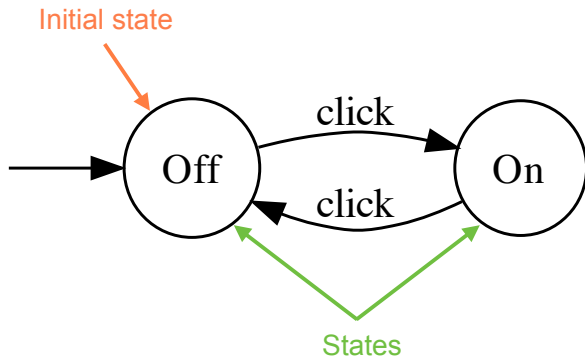
## Example: Finite Automaton



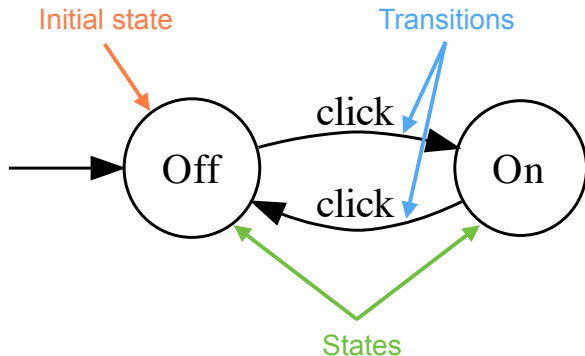
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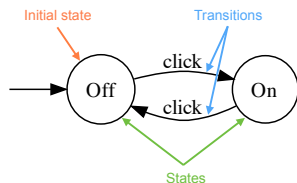
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Formally:

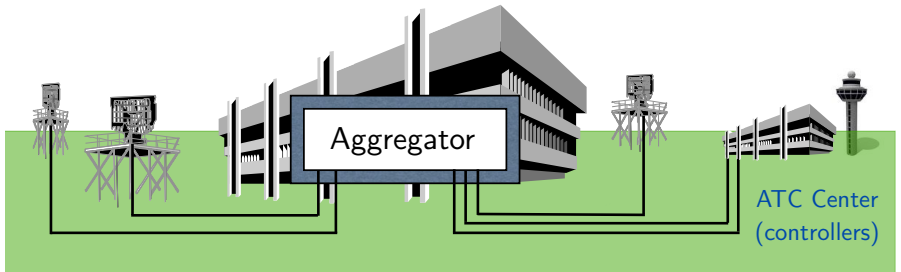
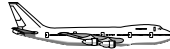
- ▶  $Q = \{\text{Off}, \text{On}\}$  is the set of **states**
- ▶  $\Sigma = \{\text{click}\}$  is the **alphabet**
- ▶ The **transition function**  $\delta$  is given by

$\delta$	click
Off	On
On	Off

- ▶ The **initial state** is Off
- ▶ There are no **accepting states**



# ATC scenario

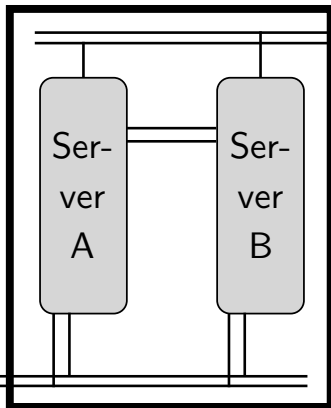




# ATC redundancy

## Active server:

- Accepts sensor data
- Provides ASP
- Sends "alive" messages



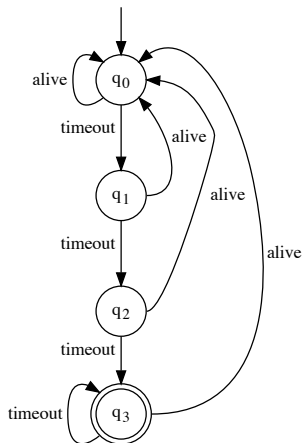
## Passive server

- Ignores sensor data
- Monitors "alive" messages
- Takes over in case of failure

Sensors

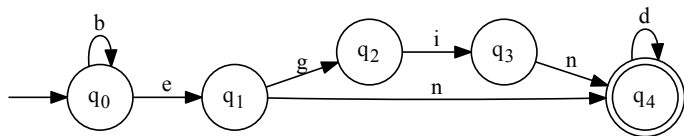
ATC

# Finite automaton to the rescue



- ▶ Two events (“letters”)
  - ▶ **timeout**: 0.1 seconds have passed
  - ▶ **alive**: message from active server
- ▶ States  $q_0, q_1, q_2$ : Server is passive
  - ▶ No processing of input
  - ▶ No sending of alive messages
- ▶ State  $q_3$ : Server becomes active
  - ▶ Process input, provide output to ATC
  - ▶ Send alive messages every 0.1 seconds

## Exercise: Finite automaton



Does this automaton accept the words *begin*, *end*, *bind*, *bend*?

## “Universal computer”

- ▶ Very simple model of a computer
  - ▶ Infinite tape, one read/write head
  - ▶ Tape can store letters from a alphabet
  - ▶ Finite automaton controls read/write and movement operations
- ▶ Very powerful model of a computer
  - ▶ Can compute anything any real computer can compute
  - ▶ Can compute anything an “ideal” real computer can compute
  - ▶ Can compute everything a human can compute (?)



# Formal grammars

Formalism to **generate** (rather than accept) words over alphabet

**terminal symbols:** may appear in the produced word (alphabet)

**non-terminal symbols:** may not appear in the produced word  
(temporary symbols)

**production rules:**  $l \rightarrow r$  means:  $l$  can be replaced by  $r$  anywhere  
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## Example

Grammar for arithmetic expressions over  $\{0, 1\}$

$$\begin{aligned}\Sigma &= \{0, 1, +, \cdot, (, )\} \\ N &= \{E\} \\ P &= \{E \rightarrow 0, E \rightarrow 1, \\ &\quad E \rightarrow (E) \\ &\quad E \rightarrow E + E \\ &\quad E \rightarrow E \cdot E\}\end{aligned}$$

# Exercise: Grammars

## Using

- ▶ the non-terminal symbols  $S, B, D, E, G, I, N$
- ▶ the terminal symbols  $b, d, e, g, i, n$
- ▶ the production rules  $S \rightarrow BEGIN,$   
 $BEG \rightarrow E, IN \rightarrow IND, IN \rightarrow N, EG \rightarrow EGG, GGG \rightarrow B,$   
 $B \rightarrow b, D \rightarrow d, E \rightarrow e, G \rightarrow g, I \rightarrow i, N \rightarrow n$

can you generate the words *bend* and *end* starting from the symbol  $S$ ?

- ▶ If yes, how many steps do you need?
- ▶ If no, why not?

# Questions about formal languages

- ▶ For a given language  $L$ , how can we find
  - ▶ a corresponding automaton  $\mathcal{A}_L$ ?
  - ▶ a corresponding grammar  $G_L$ ?
- ▶ What is the simplest automaton for  $L$ ?
  - ▶ “simplest” meaning: weakest possible language class
  - ▶ “simplest” meaning: least number of elements
- ▶ How can we use formal descriptions of languages for compilers?
  - ▶ detecting legal words/reserved words
  - ▶ testing if the structure is legal
  - ▶ understanding the meaning by analysing the structure



## More questions about formal languages

Closure properties: if  $L_1$  and  $L_2$  are in a class, does this also hold for

- ▶ the **union** of  $L_1$  and  $L_2$ ,
- ▶ the **intersection** of  $L_1$  and  $L_2$ ,
- ▶ the **concatenation** of  $L_1$  and  $L_2$ ,
- ▶ the **complement** of  $L_1$ ?

## More questions about formal languages

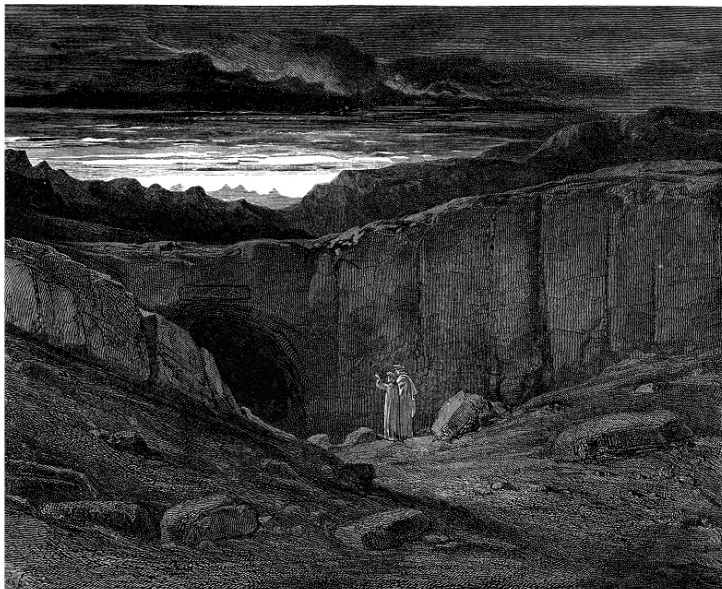
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- ▶ the **complement** of  $L_1$ ?

Decision problems: for a word  $w$  and languages  $L_1$  and  $L_2$  (given by grammars or automata),

- ▶ does  $w \in L_1$  hold?
- ▶ is  $L_1$  finite?
- ▶ is  $L_1$  empty?
- ▶ does  $L_1 = L_2$  hold?

Abandon all hope. . .



# Example applications for formal languages and automata

- ▶ HTML and web browsers
- ▶ Speech recognition and understanding grammars
- ▶ Dialog systems and AI (Siri, Alexa, *Hey Google*, Watson)
- ▶ Regular expression matching
- ▶ Compilers and interpreters of programming languages

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End lecture 1

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## Definition (Alphabet)

An **alphabet**  $\Sigma$  is a finite, non-empty set of characters (symbols, letters).

$$\Sigma = \{c_1, \dots, c_n\}$$

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## Example

- 1  $\Sigma_{\text{bin}} = \{0, 1\}$  can express integers in the binary system.
- 2 The English language is based on  $\Sigma_{\text{en}} = \{a, \dots, z, A, \dots, Z\}$ .
- 3  $\Sigma_{\text{ASCII}} = \{0, \dots, 127\}$  represents the set of ASCII characters [American Standard Code for Information Interchange] coding letters, digits, and special and control characters.



# Alphabets: ASCII code chart

ASCII Code Chart

	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	NUL	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	HT	LF	VT	FF	CR	SO	SI
1	DLE	DC1	DC2	DC3	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS	RS	US
2		!	"	#	\$	%	&	'	(	)	*	+	,	-	.	/
3	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
4	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
6	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
7	p	q	r	s	t	u	v	w	x	y	z	{		}	~	DEL

## Definition (Word)

- ▶ A **word** over the alphabet  $\Sigma$  is a finite sequence (list) of characters of  $\Sigma$ :

$$w = c_1 \dots c_n \quad \text{with} \quad c_1, \dots, c_n \in \Sigma.$$

- ▶ The **empty word** with no characters is written as  $\varepsilon$ .
- ▶ The set of all words over an alphabet  $\Sigma$  is represented by  $\Sigma^*$ .

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In programming languages, words are often referred to as **strings**.

## Example

- 1 Using  $\Sigma_{\text{bin}}$ , we can define the words  $w_1, w_2 \in \Sigma_{\text{bin}}^*$ :

$$w_1 = 01100 \quad \text{and} \quad w_2 = 11001$$

- 2 Using  $\Sigma_{\text{en}}$ , we can define the word  $w \in \Sigma_{\text{en}}^*$ :

$$w = \text{example}$$

## Definition (Length, character access)

- ▶ The **length**  $|w|$  of a word  $w$  is the number of characters in  $w$ .
- ▶ The **number of occurrences** of a character  $c$  in  $w$  is denoted as  $|w|_c$ .
- ▶ The **individual characters** within words are accessed using  $w[i]$  with  $i \in \{1, 2, \dots, |w|\}$ .

# Properties of words

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## Example

- ▶  $|\text{example}| = 7$     **and**     $|\varepsilon| = 0$
- ▶  $|\text{example}|_e = 2$     **and**     $|\text{example}|_k = 0$
- ▶  $\text{example}[4] = \text{m}$

# Appending words

## Definition (Concatenation of words)

For words  $w_1$  and  $w_2$ , the concatenation  $w_1 \cdot w_2$  is defined as  $w_1$  followed by  $w_2$ .

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$w_1 \cdot w_2$  is often simply written as  $w_1w_2$ .



# Appending words

## Definition (Concatenation of words)

For words  $w_1$  and  $w_2$ , the concatenation  $w_1 \cdot w_2$  is defined as  $w_1$  followed by  $w_2$ .

$w_1 \cdot w_2$  is often simply written as  $w_1w_2$ .

## Example

Let  $w_1 = 01$  and  $w_2 = 10$ .

Then the following holds:

$$w_1w_2 = 0110 \quad \text{and} \quad w_2w_1 = 1001$$

# Iterated concatenation

We denote the set of **natural numbers**  $\{0, 1, \dots\}$  by  $\mathbb{N}$ .

## Definition (Power of a word)

For  $n \in \mathbb{N}$ , the  **$n$ -th power**  $w^n$  of a word  $w$  concatenates the same word  $n$  times:

$$\begin{aligned}w^0 &= \varepsilon \\w^n &= w^{n-1} \cdot w \quad \text{if } n > 0\end{aligned}$$

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## Example

Let  $w = ab$ . Then:

$$\begin{aligned}w^0 &= \varepsilon \\w^1 &= ab \\w^3 &= ababab\end{aligned}$$

## Exercise: Operations on words

Given the alphabet  $\Sigma = \{a, b, c\}$  and the words

▶  $u = abc$

▶  $v = aa$

▶  $w = cb$

what is denoted by the following expressions?

1  $u^2 \cdot w$

2  $v \cdot \varepsilon \cdot w \cdot u^0$

3  $|u^3|_a$

4  $v \cdot a^2 \cdot (v[4])$

5  $(v \cdot a^2 \cdot v)[4]$

6  $|w^0|$

7  $|w^0 \cdot w|$

## Definition (Formal language)

For an alphabet  $\Sigma$ , a **formal language over  $\Sigma$**  is a subset  $L \subseteq \Sigma^*$ .

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## Example

Let  $L_{\mathbb{N}} = \{1w \mid w \in \Sigma_{\text{bin}}^*\} \cup \{0\}$ .

Then  $L_{\mathbb{N}}$  is the set of all words that represent integers using the binary system (all words starting with 1 and the word 0):

$$100 \in L_{\mathbb{N}} \quad \text{but} \quad 010 \notin L_{\mathbb{N}}.$$

## Definition (Numeric value)

We define the function

$$n : L_{\mathbb{N}} \rightarrow \mathbb{N}$$

as the function returning the numeric value of a word in the language  $L_{\mathbb{N}}$ . This means

- (a)  $n(0) = 0$ ,
- (b)  $n(1) = 1$ ,
- (c)  $n(w0) = 2 \cdot n(w)$  for  $|w| > 0$ ,
- (d)  $n(w1) = 2 \cdot n(w) + 1$  for  $|w| > 0$ .

## Definition (Prime numbers)

We define the language  $L_{\mathbb{P}}$  as the language representing prime numbers in the binary system:

$$L_{\mathbb{P}} = \{w \in L_{\mathbb{N}} \mid n(w) \in \mathbb{P}\}.$$



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One way to formally express the set of all prime numbers is

$$\mathbb{P} = \{p \in \mathbb{N}^{\geq 2} \mid \{t \in \mathbb{N} \mid \exists k \in \mathbb{N} : k \cdot t = p\} = \{1, p\}\}.$$

# C functions as a language

## Definition

We define the language  $L_C \subset \Sigma_{\text{ASCII}}^*$  as the set of all C function definitions with a declaration of the form:

$$\text{char* } f(\text{char* } x);$$

(where  $f$  and  $x$  are legal C identifiers).

Then  $L_C$  contains the ASCII code of all those definitions of C functions processing and returning a string.

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Then  $L_C$  contains the ASCII code of all those definitions of C functions processing and returning a string.

## Examples

- ▶  $\text{char* } f(\text{char* } x) \{ \text{return } x; \} \in L_C$
- ▶  $\text{char* } f(\text{char* } x) \{ \text{return } ""; \} \in L_C$
- ▶  $\text{char* } f(\text{char* } x, \text{int } y) \{ \text{return } ""; \} \notin L_C$
- ▶  $\text{Harakiri} \notin L_C$

# C function evaluations as a language

## Definition

Using the alphabet  $\Sigma_{\text{ASCII}+} = \Sigma_{\text{ASCII}} \cup \{\dagger\}$ , we define the **universal language**

$$L_u = \{f\dagger x\dagger y \mid \text{(a) and (b) and (c)}\} \quad \text{with}$$

- (a)  $f \in L_C$  (i.e.  $f$  is a C function mapping strings to strings),
- (b)  $x, y \in \Sigma_{\text{ASCII}}^*$ ,
- (c) applying  $f$  to  $x$  terminates and returns  $y$ .

# C function evaluations as a language

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## Examples

- ▶ `char* f(char* x){return x;}`†aaa†aaa  $\in L_u$
- ▶ `char* f(char* x){return x;}`†aaa†bbb  $\notin L_u$
- ▶ `char* f(char* x){return "";`†aaa†  $\in L_u$

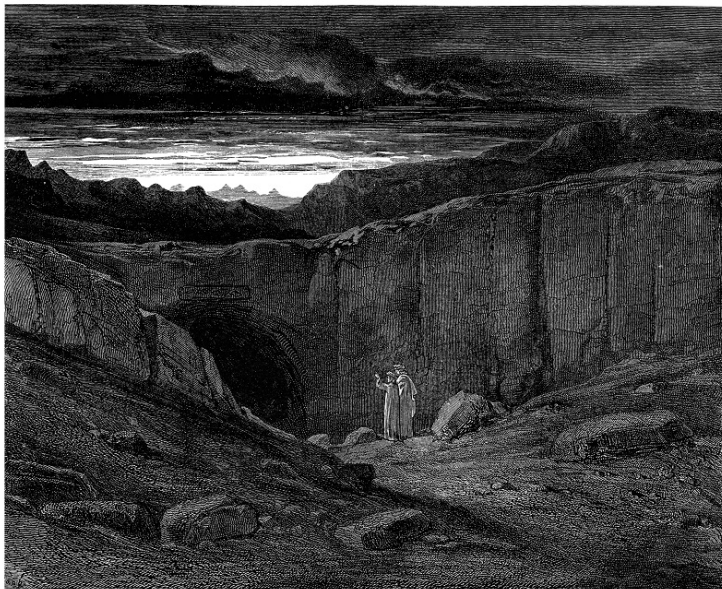
# Formal languages can be highly complex

Formal languages have a wide scope:

- ▶ Testing whether a word belongs to  $L_{\mathbb{N}}$  is straightforward
- ▶ The same test for  $L_{\mathbb{P}}$  is more complex
- ▶ The test for  $L_C$  requires a proper C parser
- ▶ Later we will see that there is no algorithm that will correctly perform the membership test for  $L_u$

```
def inN(x):  
    if x=="0":  
        return True  
    if x=="":  
        return False  
    if x[0]!="1":  
        return False  
    for c in x[1:]:  
        if c!="0" and c!="1":  
            return False  
    return True
```

Abandon all hope. . .



## Definition (Product of formal languages)

Given an alphabet  $\Sigma$  and the formal languages  $L_1, L_2 \subseteq \Sigma^*$ , we define the **product**

$$L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}.$$



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$$L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}.$$

## Example

Using the alphabet  $\Sigma_{\text{en}}$ , we define the languages

$$L_1 = \{ab, bc\} \quad \text{and} \quad L_2 = \{ac, cb\}.$$

The product is

$$L_1 \cdot L_2 = \{abac, abcb, bcac, bccb\}.$$

## Definition (Power of a language)

Given an alphabet  $\Sigma$ , a formal language  $L \subseteq \Sigma^*$ , and an integer  $n \in \mathbb{N}$ , we define the  $n$ -th power of  $L$  (recursively) as follows:

$$L^0 = \{\varepsilon\}$$

$$L^n = L^{n-1} \cdot L$$

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$$\begin{aligned}L^0 &= \{\varepsilon\} \\L^n &= L^{n-1} \cdot L\end{aligned}$$

## Example

Using the alphabet  $\Sigma_{\text{en}}$ , we define the language  $L = \{ab, ba\}$ . Thus:

$$\begin{aligned}L^0 &= \{\varepsilon\} \\L^1 &= \{\varepsilon\} \cdot \{ab, ba\} = \{ab, ba\} \\L^2 &= \{ab, ba\} \cdot \{ab, ba\} = \{abab, abba, baab, baba\}\end{aligned}$$

# The Kleene Star operator

## Definition (Kleene Star)

Given an alphabet  $\Sigma$  and a formal language  $L \subseteq \Sigma^*$ , we define the **Kleene star** operator as

$$L^* = \bigcup_{n \in \mathbb{N}} L^n.$$

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$$L^* = \bigcup_{n \in \mathbb{N}} L^n.$$

## Example

Using the alphabet  $\Sigma_{\text{en}}$ , we define the language  $L = \{a\}$ . Thus:

$$L^* = \{a^n \mid n \in \mathbb{N}\}.$$

# Exercise: formal languages

Given

- ▶ the alphabet  $\Sigma_{\text{bin}}$ ,
- ▶ the function  $n(w)$  mapping binary words to numbers as defined on page 40, and
- ▶ the language  $L = \{1\}$ ,

formally describe the following:

- a) the language  $M = L^* \setminus \{\varepsilon\}$
- b) the set  $N = \{n(w) \mid w \in M\}$
- c) the language  $M^- = \{w \mid n(w) - 1 \in N\}$
- d) the language  $M^+ = \{w \mid n(w) + 1 \in N\}$

Hinweis:  $N \neq \mathbb{N}$

# Outline

Introduction

**Regular Languages and Finite Automata**

Regular Expressions

Finite Automata

The Pumping Lemma

Properties of Regular Languages

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

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# Regular expressions

Compact and convenient way to represent a set of strings

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Compact and convenient way to represent a set of strings

- ▶ Characterize tokens for compilers
- ▶ Describe search terms for a database
- ▶ Filter through genomic data
- ▶ Extract URLs from web pages
- ▶ Extract email addresses from web pages

# Regular expressions

Compact and convenient way to represent a set of strings

- ▶ Characterize tokens for compilers
- ▶ Describe search terms for a database
- ▶ Filter through genomic data
- ▶ Extract URLs from web pages
- ▶ Extract email addresses from web pages

**The set of all regular expressions (over an alphabet)  
is a formal language**

**Each single regular expression describes a formal language**

# Introductory Examples

Consider  $\Sigma_{\text{bin}} = \{0, 1\}$ . With regular expressions we can conveniently and rigorously describe many languages:

- ▶  $1(0 + 1)^*$

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- ▶  $11^*$

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- ▶  $11^*$  – all words consisting of one or more letters 1 ( $M$  from the last exercise)
- ▶  $0(10)^* + (10)^* + 1(01)^* + (01)^*$

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- ▶  $11^*$  – all words consisting of one or more letters 1 ( $M$  from the last exercise)
- ▶  $0(10)^* + (10)^* + 1(01)^* + (01)^*$  – the language of all words where no two subsequent characters are the same

## Reminder: Power sets

### Definition (Power set of a set)

- ▶ Assume a set  $S$ . Then the power set of  $S$ , written as  $2^S$ , is the set of all subsets of  $S$ .
- ▶ In particular, if  $\Sigma$  is an alphabet,  $2^{\Sigma^*}$  is the set of all subsets of  $\Sigma^*$  and hence the set of all possible formal languages over  $\Sigma$ .



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## Example

$$2^{\Sigma_{\text{bin}}} = 2^{\{0,1\}} = \{\{\}, \{0\}, \{1\}, \{0, 1\}\},$$

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## Example

$$\begin{aligned}2^{\Sigma_{\text{bin}}} &= 2^{\{0,1\}} = \{\{\}, \{0\}, \{1\}, \{0, 1\}\}, \\2^{\Sigma_{\text{bin}}^*} &= 2^{\{\epsilon, 0, 1, 00, 01, \dots\}}\end{aligned}$$

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## Example

$$\begin{aligned}2^{\Sigma_{\text{bin}}} &= 2^{\{0,1\}} = \{\{\}, \{0\}, \{1\}, \{0, 1\}\}, \\2^{\Sigma_{\text{bin}}^*} &= 2^{\{\epsilon, 0, 1, 00, 01, \dots\}} \\&= \{\{\}, \{\epsilon\}, \{0\}, \{1\}, \{00\}, \{01\}, \dots \\&\quad \dots \{\epsilon, 0\}, \{\epsilon, 1\}, \{\epsilon, 00\}, \{\epsilon, 01\}, \dots \\&\quad \dots \{010, 1110, 10101\}, \dots\}.\end{aligned}$$

# Regular expressions and formal languages

A regular expression over  $\Sigma$ ...

- ▶ ... is a word over the extended alphabet  $\Sigma \cup \{\emptyset, \varepsilon, +, \cdot, *, (, )\}$ 
  - ▶ Note that we implicitly assume that  $\{\emptyset, \varepsilon, +, \cdot, *, (, )\} \cap \Sigma = \{\}$
  - ▶  $\emptyset$  denotes a regular expression (syntax)
  - ▶  $\{\}$  denotes the empty set (semantics)
- ▶ ... describes a formal language over  $\Sigma$

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## Terminology

The following terms are defined on the next slides:

- ▶  $R_\Sigma$  is the set of all regular expressions over the alphabet  $\Sigma$ .
- ▶ The function  $L : R_\Sigma \rightarrow 2^{\Sigma^*}$  assigns a formal language  $L(r) \subseteq \Sigma^*$  to each regular expression  $r$ .

# Regular expressions and their languages (1)

## Definition (Regular expressions)

The set of regular expressions  $R_\Sigma$  over the alphabet  $\Sigma$  is defined as follows:

- 1 The regular expression  $\emptyset$  denotes the **empty language**.  
 $\emptyset \in R_\Sigma$  and  $L(\emptyset) = \{\}$
- 2 The regular expression  $\varepsilon$  denotes the language containing only the empty word.  
 $\varepsilon \in R_\Sigma$  and  $L(\varepsilon) = \{\varepsilon\}$
- 3 Each symbol in the alphabet  $\Sigma$  is a regular expression.  
 $c \in \Sigma \Rightarrow c \in R_\Sigma$  and  $L(c) = \{c\}$

## Regular expressions and their languages (2)

### Definition (Regular expressions (cont'))

- 4 The operator  $+$  denotes the **union** of the languages of  $r_1$  and  $r_2$ .  
 $r_1 \in R_\Sigma, r_2 \in R_\Sigma \Rightarrow r_1 + r_2 \in R_\Sigma$  and  $L(r_1 + r_2) = L(r_1) \cup L(r_2)$
- 5 The operator  $\cdot$  denotes the **product** of the languages of  $r_1$  and  $r_2$ .  
 $r_1 \in R_\Sigma, r_2 \in R_\Sigma \Rightarrow r_1 \cdot r_2 \in R_\Sigma$  and  $L(r_1 \cdot r_2) = L(r_1) \cdot L(r_2)$
- 6 The **Kleene star** of a regular expression  $r$  denotes the Kleene star of the language of  $r$ .  
 $r \in R_\Sigma \Rightarrow r^* \in R_\Sigma$  and  $L(r^*) = (L(r))^*$
- 7 **Brackets** can be used to group regular expressions without changing their language.  
 $r \in R_\Sigma \Rightarrow (r) \in R_\Sigma$  and  $L((r)) = L(r)$

# Equivalence of regular expressions

## Definition (Equivalence and precedence)

- ▶ Two regular expressions  $r_1$  and  $r_2$  are **equivalent** if they denote the same language:  $r_1 \doteq r_2$  if and only if  $L(r_1) = L(r_2)$
- ▶ The operators have the following **precedence**:  
 $(\dots) > * > \cdot > +$
- ▶ The product operator  $\cdot$  can be omitted.



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## Example

$$\begin{aligned} a + b \cdot c^* &\doteq a + (b \cdot (c^*)) \\ ac + bc^* &\doteq a \cdot c + b \cdot c^* \end{aligned}$$

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## Example

$$\begin{aligned}a + b \cdot c^* &\doteq a + (b \cdot (c^*)) \\ac + bc^* &\doteq a \cdot c + b \cdot c^*\end{aligned}$$

Note: Some authors (and tools) use  $|$  as the union operator.

# Examples for regular expressions

## Example

Let  $\Sigma_{abc} = \{a, b, c\}$ .

- ▶ The regular expression  $r_1 = (a + b + c)(a + b + c)$  describes all the words of exactly two symbols:

$$L(r_1) = \{w \in \Sigma_{abc}^* \mid |w| = 2\}$$

- ▶ The regular expression  $r_2 = (a + b + c)(a + b + c)^*$  describes all the words of one or more symbols:

$$L(r_2) = \{w \in \Sigma_{abc}^* \mid |w| \geq 1\}$$

## Exercise: regular expressions

- 1 Using the alphabet  $\Sigma_{abc} = \{a, b, c\}$ , give a regular expression  $r_1$  for all the words  $w \in \Sigma_{abc}^*$  containing exactly one  $a$  or exactly one  $b$ .
- 2 Formally describe  $L(r_1)$  as a set.
- 3 Using the alphabet  $\Sigma_{abc} = \{a, b, c\}$ , give a regular expression  $r_2$  for all the words containing at least one  $a$  and at least one  $b$ .
- 4 Using the alphabet  $\Sigma_{bin} = \{0, 1\}$ , give a regular expression for all the words whose third last symbol is  $1$ .
- 5 Using the alphabet  $\Sigma_{bin}$ , give a regular expression for all the words not containing the string  $110$ .
- 6 Which language is described by the regular expression

$$r_6 = (1 + \varepsilon)(00^*1)^*0^*?$$

## Exercise: regular expressions

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- 4 Using the alphabet  $\Sigma_{bin} = \{0, 1\}$ , give a regular expression for all the words whose third last symbol is 1.
- 5 Using the alphabet  $\Sigma_{bin}$ , give a regular expression for all the words not containing the string 110.
- 6 Which language is described by the regular expression

$$r_6 = (1 + \varepsilon)(00^*1)^*0^*?$$

# Algebraic operations on regular expressions

## Theorem

1  $r_1 + r_2 \doteq r_2 + r_1$  (*commutative law*)

2  $(r_1 + r_2) + r_3 \doteq r_1 + (r_2 + r_3)$  (*associative law*)

3  $(r_1 r_2) r_3 \doteq r_1 (r_2 r_3)$  (*associative law*)

4  $\emptyset r \doteq \emptyset$

and  $r \emptyset \doteq \emptyset$

5  $\varepsilon r \doteq r$

and  $r \varepsilon \doteq r$

6  $\emptyset + r \doteq r$

7  $(r_1 + r_2) r_3 \doteq r_1 r_3 + r_2 r_3$  (*distributive law*)

8  $r_1 (r_2 + r_3) \doteq r_1 r_2 + r_1 r_3$  (*distributive law*)

## Proof of some rules

Proof of Rule 1 ( $r_1 + r_2 \doteq r_2 + r_1$ ).

$$L(r_1 + r_2) = L(r_1) \cup L(r_2) = L(r_2) \cup L(r_1) = L(r_2 + r_1)$$



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Proof of Rule 4 ( $\emptyset r \doteq \emptyset$ ).

$$L(\emptyset r) \quad \text{Def. } \underline{\text{concat}} \quad L(\emptyset) \cdot L(r)$$



## Proof of some rules

Proof of Rule 1 ( $r_1 + r_2 \doteq r_2 + r_1$ ).

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Proof of Rule 4 ( $\emptyset r \doteq \emptyset$ ).

$$\begin{array}{lcl} L(\emptyset r) & \text{Def. } \underline{\text{concat}} & L(\emptyset) \cdot L(r) \\ & \text{Def. } \underline{\text{empty regex}} & \{\} \cdot L(r) \end{array}$$

## Proof of some rules

Proof of Rule 1 ( $r_1 + r_2 \doteq r_2 + r_1$ ).

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Proof of Rule 4 ( $\emptyset r \doteq \emptyset$ ).

$$\begin{aligned} L(\emptyset r) & \stackrel{\text{Def. } \underline{\text{concat}}}{=} L(\emptyset) \cdot L(r) \\ & \stackrel{\text{Def. } \underline{\text{empty regex}}}{=} \{\} \cdot L(r) \\ & \stackrel{\text{Def. } \underline{\text{product}}}{=} \{w_1 w_2 \mid w_1 \in \{\}, w_2 \in L(r)\} \end{aligned}$$

# Proof of some rules

Proof of Rule 1 ( $r_1 + r_2 \doteq r_2 + r_1$ ).

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Proof of Rule 4 ( $\emptyset r \doteq \emptyset$ ).

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# Proof of some rules

Proof of Rule 1 ( $r_1 + r_2 \doteq r_2 + r_1$ ).

$$L(r_1 + r_2) = L(r_1) \cup L(r_2) = L(r_2) \cup L(r_1) = L(r_2 + r_1)$$



Proof of Rule 4 ( $\emptyset r \doteq \emptyset$ ).

$$\begin{aligned} L(\emptyset r) &\stackrel{\text{Def. concat}}{=} L(\emptyset) \cdot L(r) \\ &\stackrel{\text{Def. empty regexp}}{=} \{\} \cdot L(r) \\ &\stackrel{\text{Def. product}}{=} \{w_1 w_2 \mid w_1 \in \{\}, w_2 \in L(r)\} \\ &= \{\} \\ &\stackrel{\text{Def. empty regexp}}{=} L(\emptyset) \end{aligned}$$



## Theorem

9  $r + r \doteq r$

10  $(r^*)^* \doteq r^*$

11  $\emptyset^* \doteq \varepsilon$

12  $\varepsilon^* \doteq \varepsilon$

13  $r^* \doteq \varepsilon + r^*r$

14  $r^* \doteq (\varepsilon + r)^*$

15  $\varepsilon \notin L(s)$  and  $r \doteq rs + t \longrightarrow r \doteq ts^*$  (proof by Arto Salomaa)

16  $r^*r \doteq rr^*$  (see Lemma: Kleene Star below)

17  $\varepsilon \notin L(s)$  and  $r \doteq sr + t \longrightarrow r \doteq s^*t$  (Arden's Lemma)

# Lemma: Kleene Star (1)

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□

## Lemma: Kleene Star (2)

Proof of Case 2:  $\varepsilon \in L(u)$ .

We show  $L(u^*u) = L(u^*) = L(uu^*)$

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$$L(u^*u) = L(u^*) \cdot L(u)$$

$$= (L(u))^* \cdot L(u)$$

$$= \left(\bigcup_{i \geq 0} L(u)^i\right) \cdot L(u)$$

$$= \bigcup_{i \geq 0} (L(u)^i \cdot L(u))$$

$$= \bigcup_{i \geq 1} L(u)^i$$

$$\subseteq L(u^*)$$

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- ▶ a) and b) imply  $L(u^*u) = L(u^*)$
- ▶  $L(uu^*) = L(u^*)$ : strictly analogous



# A note on Arto/Arden

- ▶ Arto:  $\varepsilon \notin L(s)$  and  $r \doteq rs + t \longrightarrow r \doteq ts^*$
- ▶ Why do we need  $\varepsilon \notin L(s)$ ?
  - ▶ This guarantees that **only** words of the form  $ts^*$  are in  $L(r)$
  - ▶ Example:  $r \doteq rs + t$  mit  $s = b^*$ ,  $t = a$ .
    - ▶ If we could apply Arto, the result would be  $r \doteq a(b^*)^* \doteq ab^*$
    - ▶ But  $L = \{ab^*\} \cup \{b^*\}$  also fulfills the equation, i.e. there is no single unique solution in this case
  - ▶ Intuitively:  $\varepsilon \in L(s)$  is a second escape from the recursion that bypasses  $t$
- ▶ The case for Arden's lemma ( $\varepsilon \notin L(s)$  and  $r \doteq sr + t \longrightarrow r \doteq s^*t$ ) is analogous

## Exercise: Algebra on regular expressions

- 1 Prove the equivalence using only algebraic operations:

$$r^* \doteq \varepsilon + r^*.$$

- 2 Simplify the regular expression  $s$  using only algebraic operations:

$$s = 0(\varepsilon + 0 + 1)^* + (\varepsilon + 1)(1 + 0)^* + \varepsilon.$$

- 3 Prove the equivalence using only algebraic equivalences:

$$10(10)^* \doteq 1(01)^*0.$$



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Solutions

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Solutions

End lecture 3

# Outline

Introduction

**Regular Languages and Finite Automata**

Regular Expressions

**Finite Automata**

The Pumping Lemma

Properties of Regular Languages

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

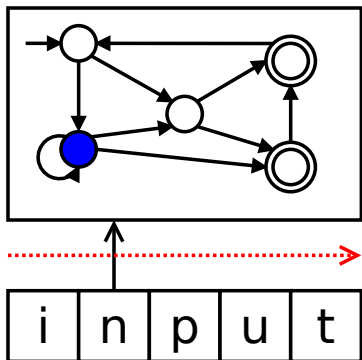
Bonus Exercises

Selected Solutions

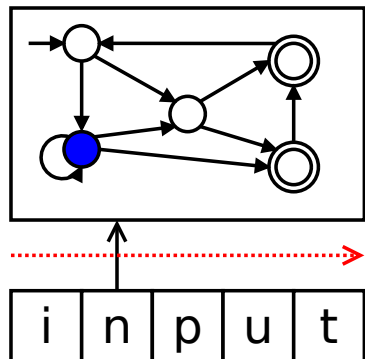
# Finite Automata: Motivation

- ▶ Simple model of computation
- ▶ Can recognize **regular languages**
- ▶ Equivalent to regular expressions
  - ▶ We can automatically generate a FA from a RE
  - ▶ We can automatically generate an RE from an FA
- ▶ Two variants:
  - ▶ Deterministic (DFA, now)
  - ▶ Non-deterministic (NFA, later)
- ▶ Easy to implement in actual programs

# Deterministic Finite Automata: Idea



# Deterministic Finite Automata: Idea



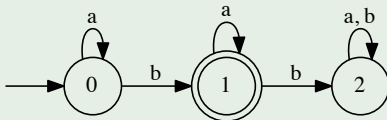
Deterministic finite automaton (DFA)

- ▶ is in one of finitely many **states**
- ▶ starts in the **initial state**
- ▶ processes **input** from left to right
  - ▶ changes state depending on character read
  - ▶ determined by **transition function**
  - ▶ no rewinding!
  - ▶ no writing!
- ▶ accepts input if
  - ▶ after reading the entire input
  - ▶ a **final state** is reached

# DFA $\mathcal{A}$ for $a^*ba^*$

## Example (Automaton $\mathcal{A}$ )

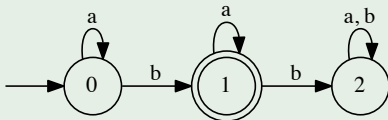
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# DFA $\mathcal{A}$ for $a^*ba^*$

## Example (Automaton $\mathcal{A}$ )

$\mathcal{A}$  is a simple DFA recognizing the regular language  $a^*ba^*$ .



- ▶  $\mathcal{A}$  has three **states**, 0, 1 and 2.
- ▶ It operates on the **alphabet**  $\{a, b\}$ .
- ▶ The **transition function** is indicated by the arrows.
- ▶ 0 is the **initial** state (with an arrow pointing at it *out of nowhere*).
- ▶ 1 is an **accepting** state (represented as a double circle).
- ▶ 2 is also called a **junk** state (once it is reached, the word can never be accepted).



# DFA: formal definition

## Definition (Deterministic Finite Automaton)

A **deterministic finite automaton** (DFA) is a quintuple  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$  with the following components

- ▶  $Q$  is a finite set of **states**.
- ▶  $\Sigma$  is the (finite) input **alphabet**.
- ▶  $\delta : Q \times \Sigma \rightarrow Q$  is the **transition function**.
- ▶  $q_0 \in Q$  is the **initial** state.
- ▶  $F \subseteq Q$  is the set of final (or **accepting**) states.

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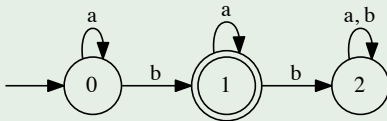
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Notes:

- ▶  $\delta$  is a **total** function – there has to be a transition from every state for every letter
- ▶ ... but automata with **partial** transition functions can be “repaired” into a proper DFA by adding a junk state

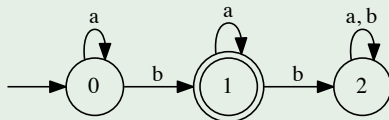
# Formal definition of $\mathcal{A}$

## Example



# Formal definition of $\mathcal{A}$

## Example



$\mathcal{A}$  is expressed as  $(Q, \Sigma, \delta, q_0, F)$  with

- ▶  $Q = \{0, 1, 2\}$
- ▶  $\Sigma = \{a, b\}$
- ▶  $\delta(0, a) = 0; \delta(0, b) = 1, \delta(1, a) = 1; \delta(1, b) = \delta(2, a) = \delta(2, b) = 2$
- ▶  $q_0 = 0$
- ▶  $F = \{1\}$

# Language accepted by an DFA

## Definition (Language accepted by an automaton)

The state transition function  $\delta$  is generalised to a function  $\delta'$  whose second argument is a word as follows:

- ▶  $\delta' : Q \times \Sigma^* \rightarrow Q$
- ▶  $\delta'(q, \varepsilon) = q$  for every  $q \in Q$
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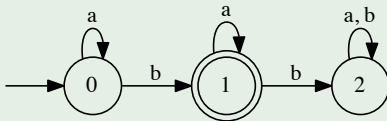
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The **language accepted by a DFA**  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$  is defined as

$$L(\mathcal{A}) = \{w \in \Sigma^* \mid \delta'(q_0, w) \in F\}.$$

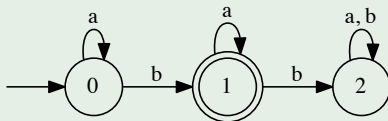
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## Example



# Language accepted by $\mathcal{A}$

## Example



- ▶  $\delta'(0, aa) = \delta(\delta'(0, a), a) = \delta(\delta(\delta'(0, \varepsilon), a), a) = 0$
- ▶  $\delta'(1, aaa) = 1$
- ▶  $\delta'(0, bb) = \delta'(1, b) = 2$
- ▶  $L(\mathcal{A}) = \{w \in \{a, b\}^* \mid w = a^n b a^m \text{ and } n, m \in \mathbb{N}\}$



# Run of a DFA

## Definition (Configuration, Run)

Let  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$  be a DFA.

A **configuration** of  $\mathcal{A}$  is a pair  $(q, w)$  with  $q \in Q$  and  $w \in \Sigma^*$ .

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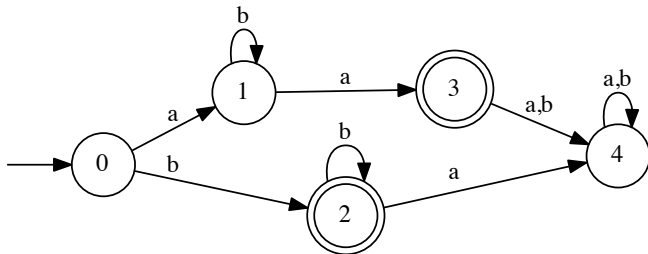
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A run is **accepting** if  $q_n \in F$  holds.

The language accepted by  $\mathcal{A}$  can alternatively be defined as the set of all words for which there exists an accepting run of  $\mathcal{A}$ .

# Exercise: DFA

1 Given this graphical representation of a DFA  $\mathcal{A}$ :



- Give a regular expression describing  $L(\mathcal{A})$ .
- Give a formal definition of  $\mathcal{A}$ .

# Exercise: DFA

## 2 Give

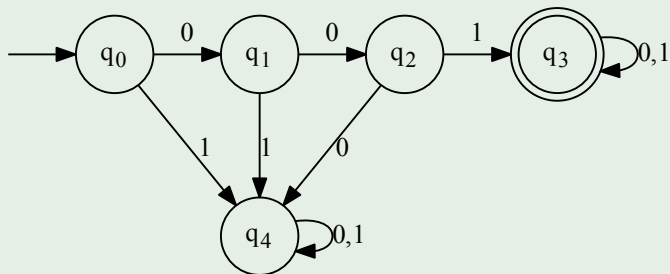
- ▶ a regular expression,
- ▶ a graphical representation of a DFA, and
- ▶ a formal definition

for the language  $L \subset \{a, b\}^*$  containing all those words featuring the substring  $ab$

- a) at the beginning,
- b) at an arbitrary position,
- c) at the end.

# Another example

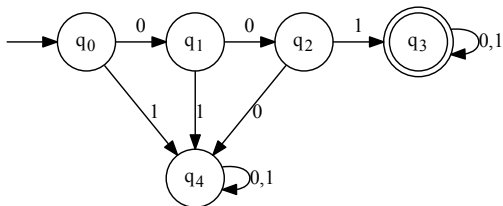
## Example



Which language is recognized by the DFA?



# Tabular representation of a DFA

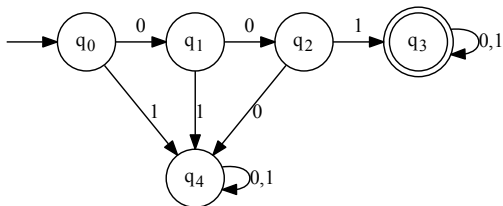


$$\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$$

- ▶  $Q = \{q_0, q_1, q_2, q_3, q_4\}$
- ▶  $\Sigma = \{0, 1\}$
- ▶ Initial state:  $q_0$
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$\delta$	0	1
$q_0$	$q_1$	$q_4$
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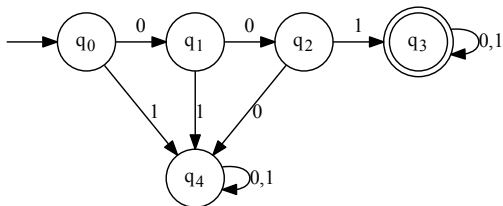


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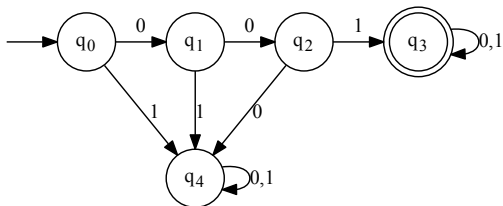


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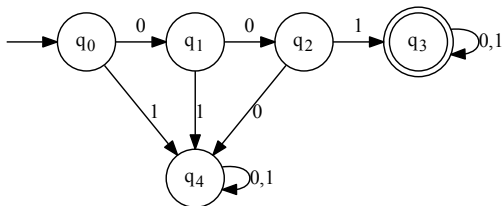


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	$q_2$	$q_4$	$q_3$
*	$q_3$	$q_3$	$q_3$
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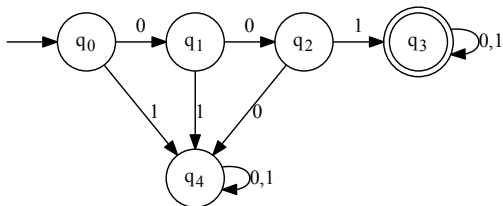
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▶ Initial state:  $q_0$

▶  $F = \{q_3\}$

		$\delta$	0	1
$\rightarrow$	$q_0$	$q_1$	$q_4$	
	$q_1$	$q_2$	$q_4$	
	$q_2$	$q_4$	$q_3$	
*	$q_3$	$q_3$	$q_3$	
	$q_4$	$q_4$	$q_4$	

# DFA: Tabular representation in practice

Delta		0	1
-> q0		q1	q4
q1		q2	q4
q2		q4	q3
* q3		q3	q3
q4		q4	q4

# DFA: Tabular representation in practice

Delta		0	1
-----			
-> q0		q1	q4
q1		q2	q4
q2		q4	q3
* q3		q3	q3
q4		q4	q4

```
> easim.py fsa001.txt 10101
Processing: 10101
q0 :: 1 -> q4
q4 :: 0 -> q4
q4 :: 1 -> q4
q4 :: 0 -> q4
q4 :: 1 -> q4
Rejected
```



# DFA: Tabular representation in practice

Delta		0	1
-----			
-> q0		q1	q4
q1		q2	q4
q2		q4	q3
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q4		q4	q4

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Processing: 10101
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q4 :: 0 -> q4
q4 :: 1 -> q4
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```

```
> easim.py fsa001.txt 101
Processing: 101
q0 :: 1 -> q4
q4 :: 0 -> q4
q4 :: 1 -> q4
Rejected
```

# DFAs in tabular form: exercise

- ▶ Give the following DFA ...
  - ▶ as a formal 5-tuple
  - ▶ as a diagram

parity		0	1
-----			
-> even		even	odd
* odd		odd	even

- ▶ Characterize the language accepted by the DFA

▶ Assume

▶  $\Sigma = \{a, b, c\}$

▶  $L_1 = \{ubw \mid u \in \Sigma^*, w \in \Sigma\}$

▶  $L_2 = \{ubw \mid u \in \Sigma, w \in \Sigma^*\}$

▶ Group 1 (your family name starts with A-M):

Find a DFA  $\mathcal{A}$  with  $L(\mathcal{A}) = L_1$

▶ Group 2 (your family name does not start with A-M):

Find a DFA  $\mathcal{A}$  with  $L(\mathcal{A}) = L_2$

# Outline

Introduction

Regular Languages and Finite Automata

Regular Expressions

**Finite Automata**

Non-Determinism

Regular expressions and Finite Automata

Minimisation

Equivalence

The Pumping Lemma

Properties of Regular Languages

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

# Drawbacks of deterministic automata

Deterministic automata:

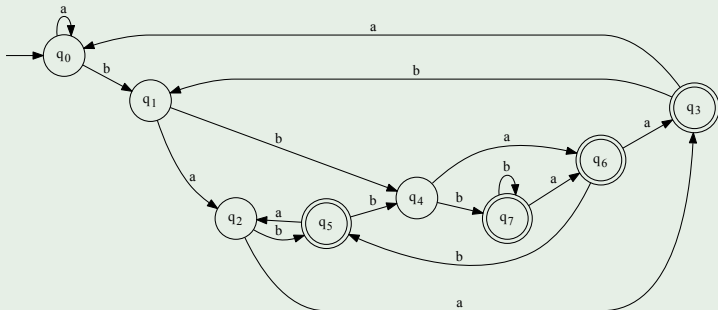
- ▶ Transition function  $\delta$ 
  - ▶ exactly one transition from every configuration
- ▶ can be complex even for simple languages

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Example (DFA  $\mathcal{A}$  for  $(a + b)^*b(a + b)(a + b)$ )



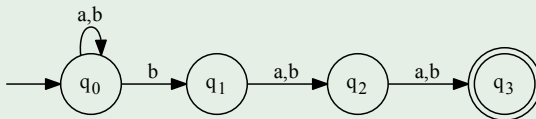
# Non-Determinism

- ▶ FA can be simplified if one input can lead to
  - ▶ one transition,
  - ▶ multiple transitions, or
  - ▶ no transition.
- ▶ Intuitively, such an FA selects its next state from a set of states depending on the current state and the input
  - ▶ and always chooses the “right” one
- ▶ This is called a **non-deterministic finite automaton** (NFA)

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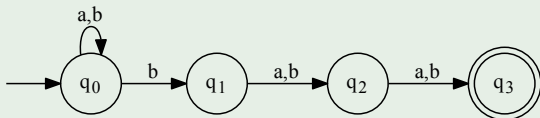
Example (NFA  $\mathcal{B}$  for  $(a + b)^*b(a + b)(a + b)$ )





# Non-Deterministic automata

## Example (Transitions in $\mathcal{B}$ )



- ▶ In state  $q_0$  with input  $b$ , the FA has to “guess” the next state.
- ▶ The string  $abab$  can be read in three ways:
  - 1  $q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_0$  (failure)
  - 2  $q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_1$  (failure)
  - 3  $q_0 \xrightarrow{a} q_0 \xrightarrow{b} q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_3$  (success)
- ▶ An NFA accepts an input  $w$  if there **exists** an accepting run on  $w$ !

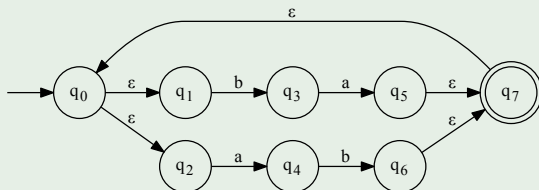
# NFA: non-deterministic transitions and $\varepsilon$ -transitions

- ▶ Non-deterministic transitions allow an NFA to go to more than one successor state
  - ▶ Instead of a **function**  $\delta$ , an NFA has a transition **relation**  $\Delta$
- ▶ An NFA can additionally change its current state without reading an input symbol:  $q_1 \xrightarrow{\varepsilon} q_2$ .
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## Example (NFA with $\epsilon$ -transitions)



## Definition (NFA)

An **NFA** is a quintuple  $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$  with the following components:

- 1  $Q$  is the finite set of states.
- 2  $\Sigma$  is the input alphabet.
- 3  $\Delta$  is a **relation** over  $Q \times (\Sigma \cup \{\varepsilon\}) \times Q$ .
- 4  $q_0 \in Q$  is the initial state.
- 5  $F \subseteq Q$  is the set of final states.

# Run of a nondeterministic automaton

## Definition (Configuration, Run of an NFA)

Let  $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$  be an NFA.

A **configuration** of  $\mathcal{A}$  is a pair  $(q, w)$  as for a DFA.

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$$r = ((q_0, w_0), (q_1, w_1), \dots, (q_m, \varepsilon))$$

such that the following conditions are satisfied:

- ▶  $q_i \in Q$  for all  $1 \leq i \leq m$ ,
- ▶ if  $r$  contains the configurations  $(q_i, w_i), (q_{i+1}, w_{i+1})$ , then
  - ▶ there is a transition  $(q_i, c, q_{i+1}) \in \Delta$  with  $c \in \Sigma \cup \{\varepsilon\}$
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The slightly more complex definition is necessary to handle  $\varepsilon$ -transitions.



# Language recognized by an NFA

## Definition (Language recognized by an NFA)

Let  $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$  be an NFA. The language accepted by  $\mathcal{A}$  is

$$L(\mathcal{A}) = \{w \mid \text{there is an accepting run of } \mathcal{A} \text{ on } w\}$$

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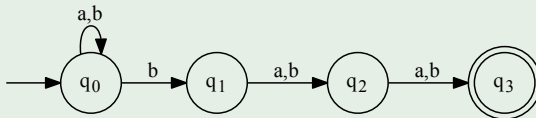
$$L(\mathcal{A}) = \{w \mid \text{there is an accepting run of } \mathcal{A} \text{ on } w\}$$

Note:

- ▶ Only existence of one accepting run is required
- ▶ It does not matter if there are also non-accepting runs on  $w$

# Example: NFA definition

## Example (Formal definition of $\mathcal{B}$ )



$\mathcal{B} = (Q, \Sigma, \Delta, q_0, F)$  with

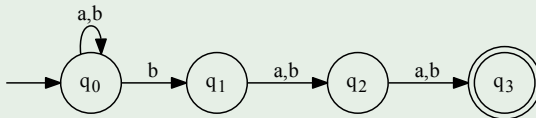
$Q = \{q_0, q_1, q_2, q_3\}$

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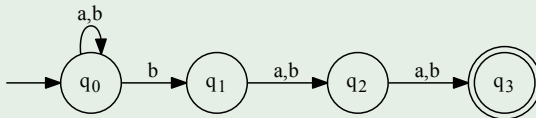
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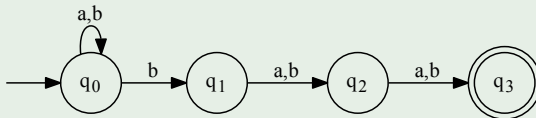
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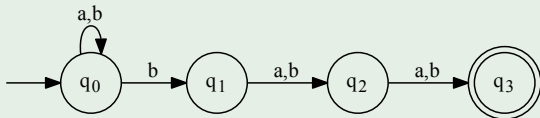
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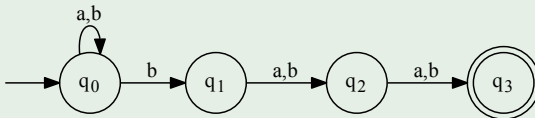
$\Sigma = \{a, b\}$

$F = \{q_3\}$

$\Delta = \{(q_0, a, q_0), (q_0, b, q_0), (q_0, b, q_1),$   
 $(q_1, a, q_2), (q_1, b, q_2),$   
 $(q_2, a, q_3), (q_2, b, q_3)\}$

# Example: NFA definition

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$\mathcal{B} = (Q, \Sigma, \Delta, q_0, F)$  with

$Q = \{q_0, q_1, q_2, q_3\}$

$\Sigma = \{a, b\}$

$F = \{q_3\}$

$\Delta$	a	b	$\epsilon$
$q_0$	$\{q_0\}$	$\{q_0, q_1\}$	$\{\}$
$q_1$	$\{q_2\}$	$\{q_2\}$	$\{\}$
$q_2$	$\{q_3\}$	$\{q_3\}$	$\{\}$
$q_3$	$\{\}$	$\{\}$	$\{\}$



## Exercise: NFA

Develop an NFA  $\mathcal{A}$  whose language  $L(\mathcal{A}) \subset \{a, b\}^*$  contains all those words featuring the substring  $aba$ . Give:

- ▶ a regular expression representing  $L(\mathcal{A})$ ,
- ▶ a graphical representation of  $\mathcal{A}$ ,
- ▶ a formal definition of  $\mathcal{A}$ .

# Equivalence of DFA and NFA

## Theorem (Equivalence of DFA and NFA)

*NFAs and DFAs recognize the same class of languages.*

- ▶ *For every DFA  $\mathcal{A}$  there is an NFA  $\mathcal{B}$  with  $L(\mathcal{A}) = L(\mathcal{B})$ .*
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- ▶ The direction DFA to NFA is trivial:
  - ▶ Every DFA is (essentially) an NFA
  - ▶ ... since every function is a relation
- ▶ What about the other direction?

# Equivalence of DFA and NFA

Equivalence of DFAs and NFAs can be shown by transforming

- ▶ an NFA  $\mathcal{A}$
- ▶ into a DFA  $\text{det}(\mathcal{A})$  accepting the same language.

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Method:

- ▶ states of  $\text{det}(\mathcal{A})$  represent **sets of states** of  $\mathcal{A}$
- ▶ a transition from  $q_1$  to  $q_2$  with character  $c$  in  $\text{det}(\mathcal{A})$  is possible if
  - ▶ in  $\mathcal{A}$  there is a transition with  $c$
  - ▶ from **one** of the states that  $q_1$  represents
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To this end, we define three auxiliary functions.

- ▶  $ec$  to compute the  $\varepsilon$  closure of a state
- ▶  $\delta^*$  to compute possible successors of a state
- ▶  $\hat{\delta}$ , the new transition function for the generated DFA

## Step 1: $\epsilon$ closure of an NFA

The  $\epsilon$  closure of a state  $q$  contains all states the NFA can change to by means of  $\epsilon$  transitions starting from  $q$ .

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### Definition ( $\varepsilon$ closure)

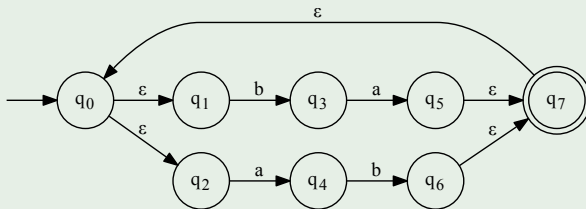
The function  $ec : Q \rightarrow 2^Q$  is the smallest function with the properties:

- ▶  $q \in ec(q)$
- ▶  $p \in ec(q) \wedge (p, \varepsilon, r) \in \Delta \Rightarrow r \in ec(q)$



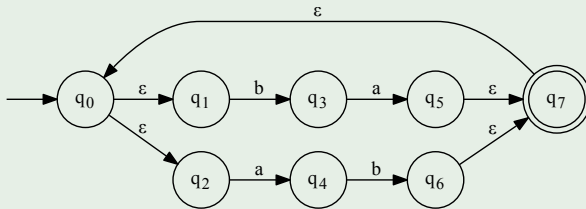
# Example: $\epsilon$ closure

## Example



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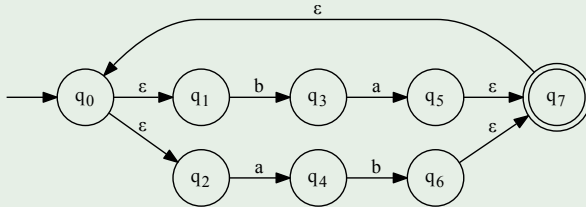
## Example



►  $ec(q_0) =$

# Example: $\epsilon$ closure

## Example

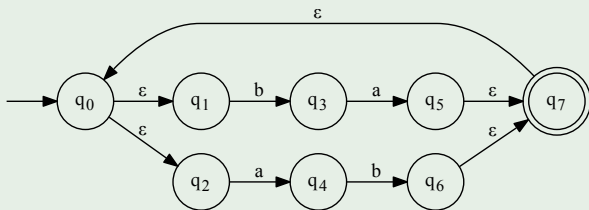


▶  $ec(q_0) = \{q_0, q_1, q_2\}$ ,

▶  $ec(q_1) =$

# Example: $\epsilon$ closure

## Example



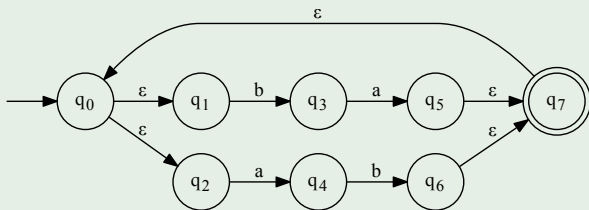
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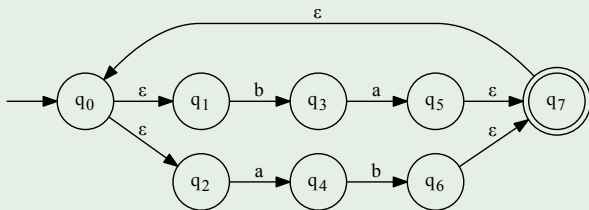
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- ▶  $ec(q_3) =$

# Example: $\epsilon$ closure

## Example



▶  $ec(q_0) = \{q_0, q_1, q_2\}$ ,

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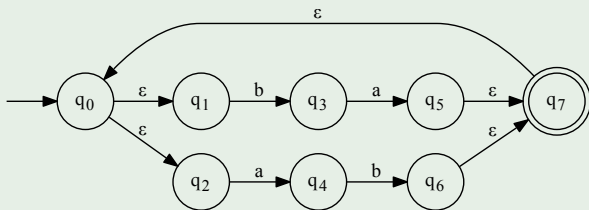
▶  $ec(q_2) = \{q_2\}$ ,

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▶  $ec(q_4) =$

# Example: $\epsilon$ closure

## Example



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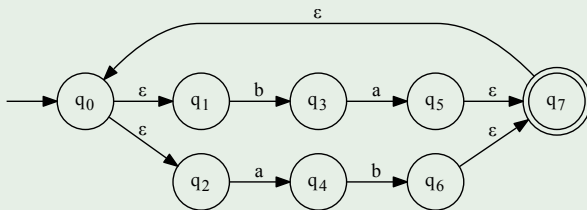
▶  $ec(q_3) = \{q_3\}$ ,

▶  $ec(q_4) = \{q_4\}$ ,

▶  $ec(q_5) =$

# Example: $\epsilon$ closure

## Example



▶  $ec(q_0) = \{q_0, q_1, q_2\}$ ,

▶  $ec(q_1) = \{q_1\}$ ,

▶  $ec(q_2) = \{q_2\}$ ,

▶  $ec(q_3) = \{q_3\}$ ,

▶  $ec(q_4) = \{q_4\}$ ,

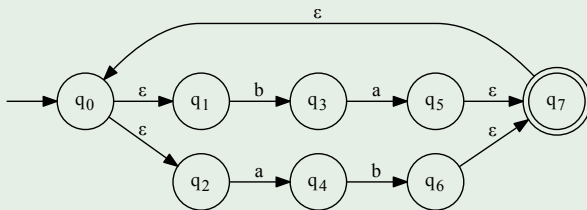
▶  $ec(q_5) = \{q_5, q_7, q_0, q_1, q_2\}$ ,

▶  $ec(q_6) =$



# Example: $\epsilon$ closure

## Example



▶  $ec(q_0) = \{q_0, q_1, q_2\}$ ,

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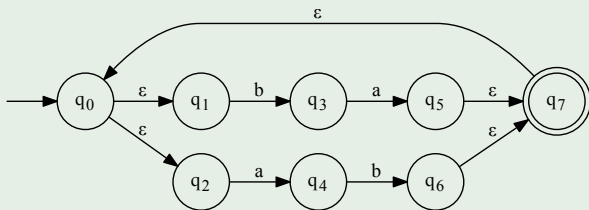
▶  $ec(q_5) = \{q_5, q_7, q_0, q_1, q_2\}$ ,

▶  $ec(q_6) = \{q_6, q_7, q_0, q_1, q_2\}$ ,

▶  $ec(q_7) =$

# Example: $\epsilon$ closure

## Example



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▶  $ec(q_1) = \{q_1\}$ ,

▶  $ec(q_2) = \{q_2\}$ ,

▶  $ec(q_3) = \{q_3\}$ ,

▶  $ec(q_4) = \{q_4\}$ ,

▶  $ec(q_5) = \{q_5, q_7, q_0, q_1, q_2\}$ ,

▶  $ec(q_6) = \{q_6, q_7, q_0, q_1, q_2\}$ ,

▶  $ec(q_7) = \{q_7, q_0, q_1, q_2\}$ .

## Step 2: Successor state function for NFAs

The function  $\delta^*$  maps

- ▶ a pair  $(q, c)$  (state and character)
- ▶ to the set of **all** states the NFA can change to from  $q$  with  $c$
- ▶ followed by any number of  $\varepsilon$  transitions.

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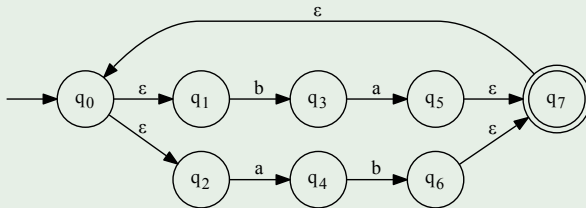
### Definition (Successor state function)

The function  $\delta^* : Q \times \Sigma \rightarrow 2^Q$  is defined as follows:

$$\delta^*(q, c) = \bigcup_{r \in Q: (q, c, r) \in \Delta} ec(r)$$

# Example: successor state function

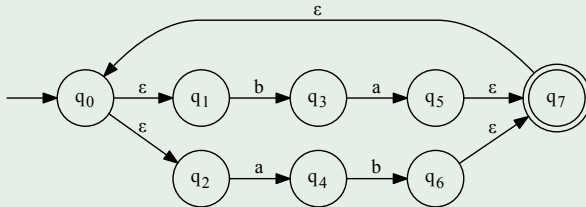
## Example



$$\delta^*(q, c) = \bigcup_{r \in Q: (q, c, r) \in \Delta} ec(r)$$

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## Example

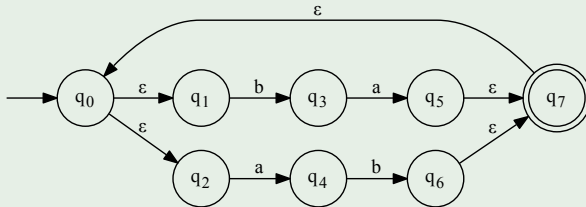


$$\delta^*(q, c) = \bigcup_{r \in Q: (q, c, r) \in \Delta} ec(r)$$

►  $\delta^*(q_0, a) =$

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## Example

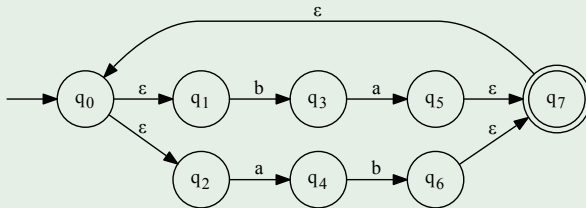


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- ▶  $\delta^*(q_1, b) =$

# Example: successor state function

## Example



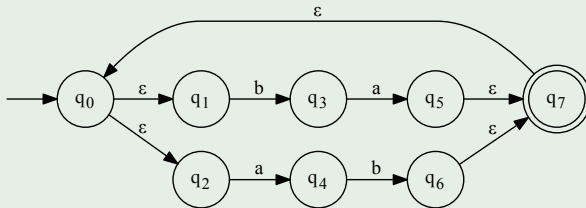
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# Example: successor state function

## Example



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- ▶  $\delta^*(q_0, a) = \{\}$ ,
- ▶  $\delta^*(q_1, b) = \{q_3\}$ ,
- ▶  $\delta^*(q_3, a) = \{q_5, q_7, q_0, q_1, q_2\}$ ,
- ▶ ...

## Step 3: extended transition function

The function  $\hat{\delta}$  maps

- ▶ a pair  $(M, c)$  consisting of a **set** of states  $M$  and a character  $c$
- ▶ to the **set**  $N$  of states that are reachable from **any** state of  $M$  via  $\Delta$  by reading the character  $c$
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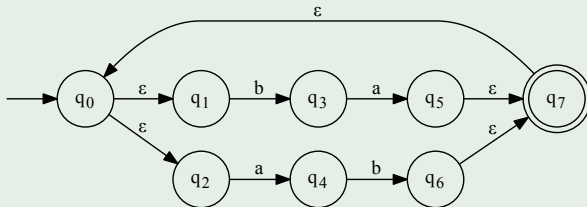
### Definition (Extended transition function)

The function  $\hat{\delta} : 2^Q \times \Sigma \rightarrow 2^Q$  is defined as follows:

$$\hat{\delta}(M, c) = \bigcup_{q \in M} \delta^*(q, c).$$

# Example: extended transition function

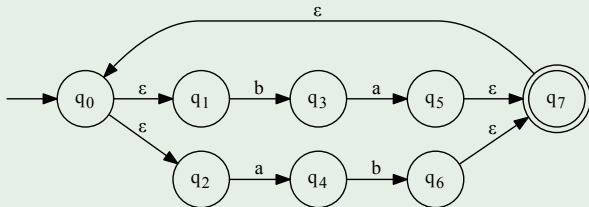
## Example



- ▶  $\delta^*(q_0, a) = \{\}$
- ▶  $\delta^*(q_1, b) = \{q_3\}$
- ▶  $\delta^*(q_3, a) = \{q_5, q_7, q_0, q_1, q_2\}$
- ▶ ...

# Example: extended transition function

## Example



▶  $\delta^*(q_0, a) = \{\}$

▶  $\delta^*(q_1, b) = \{q_3\}$

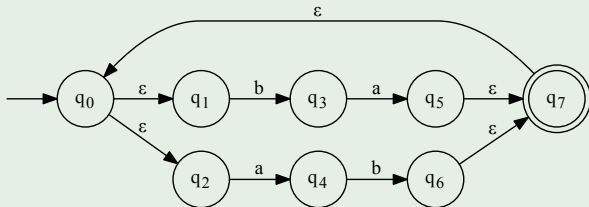
▶  $\delta^*(q_3, a) = \{q_5, q_7, q_0, q_1, q_2\}$

▶ ...

▶  $\hat{\delta}(\{q_0, q_1, q_2\}, a) =$

# Example: extended transition function

## Example



▶  $\delta^*(q_0, a) = \{\}$

▶  $\delta^*(q_1, b) = \{q_3\}$

▶  $\delta^*(q_3, a) = \{q_5, q_7, q_0, q_1, q_2\}$

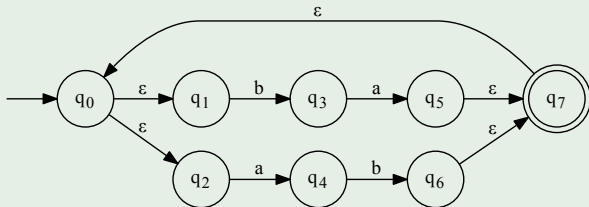
▶ ...

▶  $\hat{\delta}(\{q_0, q_1, q_2\}, a) = \{q_4\}$

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# Example: extended transition function

## Example



▶  $\delta^*(q_0, a) = \{\}$

▶  $\delta^*(q_1, b) = \{q_3\}$

▶  $\delta^*(q_3, a) = \{q_5, q_7, q_0, q_1, q_2\}$

▶ ...

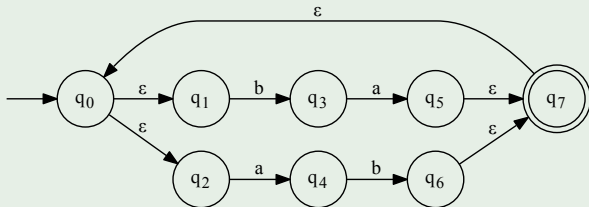
▶  $\hat{\delta}(\{q_0, q_1, q_2\}, a) = \{q_4\}$

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▶  $\delta^*(q_0, a) = \{\}$

▶  $\delta^*(q_1, b) = \{q_3\}$

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▶  $\hat{\delta}(\{q_3\}, a) = \{q_5, q_7, q_0, q_1, q_2\}$

▶  $\hat{\delta}(\{q_3\}, b) = \{\}$

▶ ...



## Equivalence of DFA and NFA: formal definition

Using the auxiliary functions  $ec$ ,  $\hat{\delta}$ , we can define  $\text{det}(\mathcal{A})$ .

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For an NFA  $\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$ , the **deterministic** Automaton  $\det(\mathcal{A})$  is defined as

$$(2^Q, \Sigma, \hat{\delta}, ec(q_0), \hat{F})$$

with  $\hat{F} = \{M \in 2^Q \mid M \cap F \neq \{\}\}$ .

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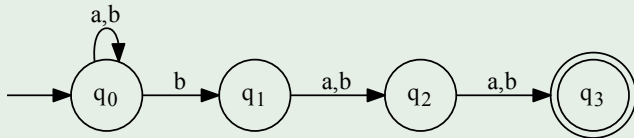
The set of final states  $\hat{F}$  is the set of all subsets of  $Q$  containing a final state.

## Remark

In practice, we use a more efficient stepwise construction that only builds the *reachable* states, not all of  $2^Q$ !

# Example: transformation into DFA

Example (NFA  $\mathcal{B}$  for  $(a + b)^*b(a + b)(a + b)$ )

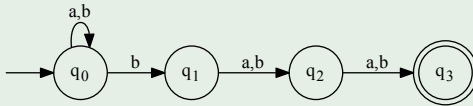


$$\begin{aligned}\mathcal{B} &= (\{q_0, q_1, q_2, q_3\}, \{a, b\}, \Delta, q_0, \{q_3\}) \\ \text{det}(\mathcal{B}) &= (\hat{Q}, \{a, b\}, \hat{\delta}, S_0, \hat{F})\end{aligned}$$

► Initial state:  $S_0 := ec(q_0) = \{q_0\}$

# Example: transformation into DFA (cont')

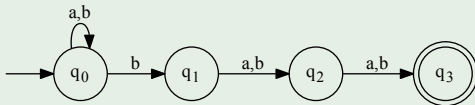
## Example



►  $\hat{\delta}(S_0, a) =$

# Example: transformation into DFA (cont')

## Example

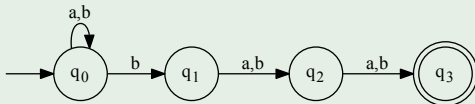


▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$

▶  $\hat{\delta}(S_0, b) =$

# Example: transformation into DFA (cont')

## Example

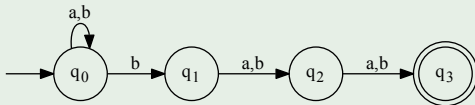


- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶  $\hat{\delta}(S_1, a) =$



# Example: transformation into DFA (cont')

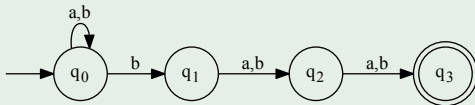
## Example



- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶  $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶  $\hat{\delta}(S_1, b) =$

# Example: transformation into DFA (cont')

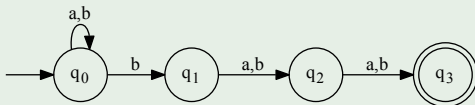
## Example



- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶  $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶  $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶  $\hat{\delta}(S_2, a) =$

# Example: transformation into DFA (cont')

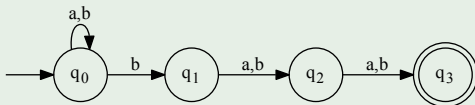
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- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶  $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶  $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶  $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
- ▶  $\hat{\delta}(S_2, b) =$

# Example: transformation into DFA (cont')

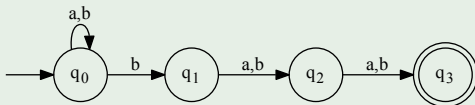
## Example



- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶  $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶  $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶  $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
- ▶  $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
- ▶  $\hat{\delta}(S_4, a) =$

# Example: transformation into DFA (cont')

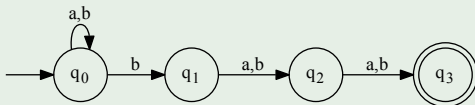
## Example



- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
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- ▶  $\hat{\delta}(S_4, a) = \{q_0, q_2, q_3\} =: S_6$
- ▶  $\hat{\delta}(S_4, b) =$

# Example: transformation into DFA (cont')

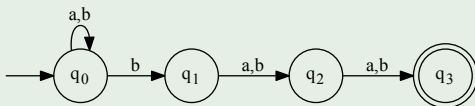
## Example



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- ▶  $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
- ▶  $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
- ▶  $\hat{\delta}(S_4, a) = \{q_0, q_2, q_3\} =: S_6$
- ▶  $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$

# Example: transformation into DFA (cont')

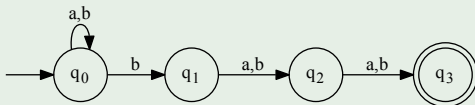
## Example



- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
  - ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
  - ▶  $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
  - ▶  $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
  - ▶  $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
  - ▶  $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
  - ▶  $\hat{\delta}(S_4, a) = \{q_0, q_2, q_3\} =: S_6$
  - ▶  $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$
- ▶  $\hat{\delta}(S_3, a) =$

# Example: transformation into DFA (cont')

## Example

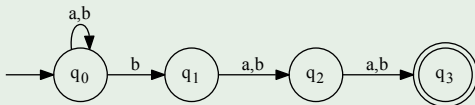


- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶  $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
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- ▶  $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$
- ▶  $\hat{\delta}(S_3, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_3, b) =$



# Example: transformation into DFA (cont')

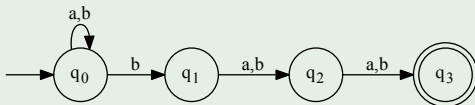
## Example



- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶  $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶  $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶  $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
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- ▶  $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$
- ▶  $\hat{\delta}(S_3, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_3, b) = \{q_0, q_1\} = S_1$
- ▶  $\hat{\delta}(S_5, a) =$

# Example: transformation into DFA (cont')

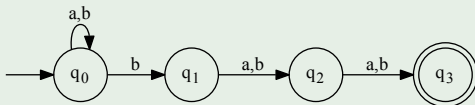
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- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶  $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶  $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶  $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
- ▶  $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
- ▶  $\hat{\delta}(S_4, a) = \{q_0, q_2, q_3\} =: S_6$
- ▶  $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$
- ▶  $\hat{\delta}(S_3, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_3, b) = \{q_0, q_1\} = S_1$
- ▶  $\hat{\delta}(S_5, a) = \{q_0, q_2\} = S_2$
- ▶  $\hat{\delta}(S_5, b) =$

# Example: transformation into DFA (cont')

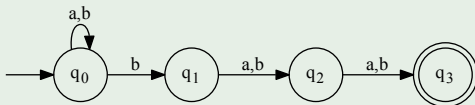
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- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
- ▶  $\hat{\delta}(S_1, a) = \{q_0, q_2\} =: S_2$
- ▶  $\hat{\delta}(S_1, b) = \{q_0, q_1, q_2\} =: S_4$
- ▶  $\hat{\delta}(S_2, a) = \{q_0, q_3\} =: S_3$
- ▶  $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
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- ▶  $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$
- ▶  $\hat{\delta}(S_3, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_3, b) = \{q_0, q_1\} = S_1$
- ▶  $\hat{\delta}(S_5, a) = \{q_0, q_2\} = S_2$
- ▶  $\hat{\delta}(S_5, b) = \{q_0, q_1, q_2\} = S_4$
- ▶  $\hat{\delta}(S_6, a) =$

# Example: transformation into DFA (cont')

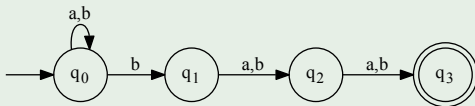
## Example



- ▶  $\hat{\delta}(S_0, a) = \{q_0\} = S_0$
- ▶  $\hat{\delta}(S_0, b) = \{q_0, q_1\} =: S_1$
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- ▶  $\hat{\delta}(S_2, b) = \{q_0, q_1, q_3\} =: S_5$
- ▶  $\hat{\delta}(S_4, a) = \{q_0, q_2, q_3\} =: S_6$
- ▶  $\hat{\delta}(S_4, b) = \{q_0, q_1, q_2, q_3\} =: S_7$
- ▶  $\hat{\delta}(S_3, a) = \{q_0\} = S_0$
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- ▶  $\hat{\delta}(S_6, a) = \{q_0, q_3\} = S_3$
- ▶  $\hat{\delta}(S_6, b) =$

# Example: transformation into DFA (cont')

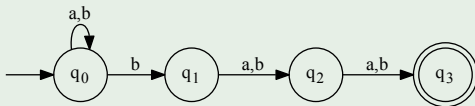
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- ▶  $\hat{\delta}(S_7, a) =$

# Example: transformation into DFA (cont')

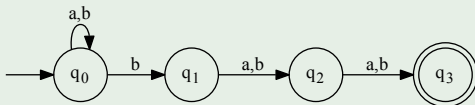
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# Example: transformation into DFA (cont')

## Example



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# Example: transformation into DFA (cont')

## Example

We can now define the DFA  $\text{det}(\mathcal{B}) = (\hat{Q}, \Sigma, \hat{\delta}, S_0, \hat{F})$  as follows:

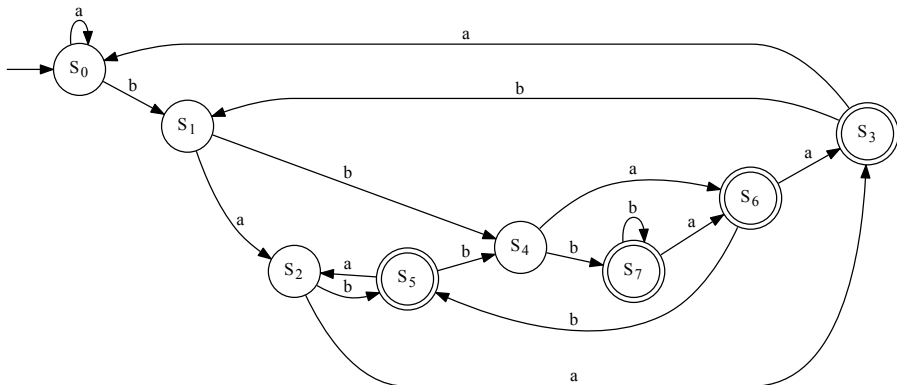
- ▶ the set of states  $\hat{Q} = \{S_0, \dots, S_7\}$ ,
- ▶ the state transition function  $\hat{\delta}$  is:

$\hat{\delta}$	$S_0$	$S_1$	$S_2$	$S_3$	$S_4$	$S_5$	$S_6$	$S_7$
a	$S_0$	$S_2$	$S_3$	$S_0$	$S_6$	$S_2$	$S_3$	$S_6$
b	$S_1$	$S_4$	$S_5$	$S_1$	$S_7$	$S_4$	$S_5$	$S_7$

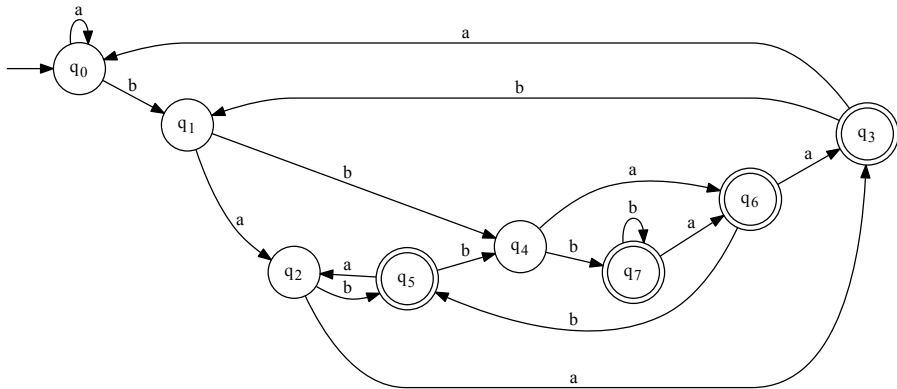
- ▶ and the set of final states  $\hat{F} = \{S_3, S_5, S_6, S_7\}$ .



# Example: transformation into DFA (cont')

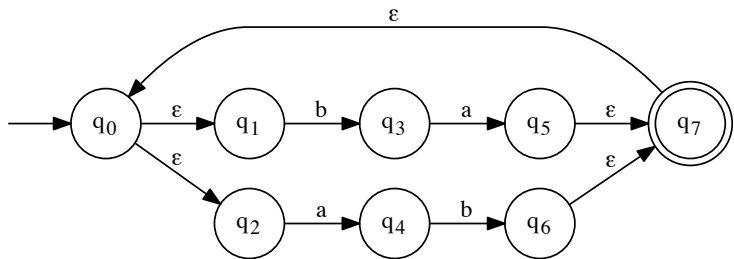


# Example: transformation into DFA (cont')



# Exercise: Transformation into DFA

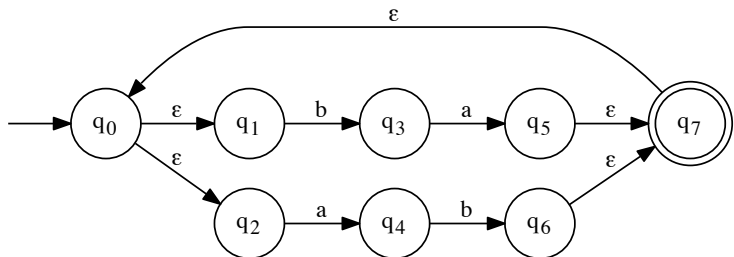
Given the following NFA  $\mathcal{A}$ :



- Determine  $\text{det}(\mathcal{A})$ .
- Draw  $\text{det}(\mathcal{A})$ 's graphical representation
- Give a regular expression representing the same language as  $\mathcal{A}$ .

# Exercise: Transformation into DFA

Given the following NFA  $\mathcal{A}$ :

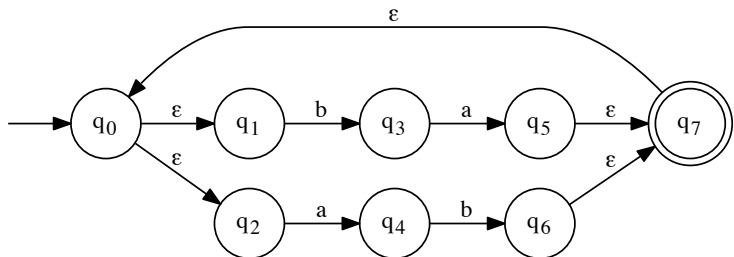


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Solution

# Exercise: Transformation into DFA

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Solution

End lecture 5

Introduction

Regular Languages and Finite Automata

Regular Expressions

**Finite Automata**

Non-Determinism

Regular expressions and Finite Automata

Minimisation

Equivalence

The Pumping Lemma

Properties of Regular Languages

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

# Regular expressions and Finite Automata

- ▶ Regular expressions describe **regular languages**
  - ▶ For each regular language  $L$ , there is an regular expression  $r$  with  $L(r) = L$
  - ▶ For every regular expression  $r$ ,  $L(r)$  is a regular language
- ▶ Finite automata describe **regular languages**
  - ▶ For each regular language  $L$ , there is a FA  $\mathcal{A}$  with  $L(\mathcal{A}) = L$
  - ▶ For every finite automaton  $\mathcal{A}$ ,  $L(\mathcal{A})$  is a regular language
- ▶ Now: constructive proof of equivalence between REs and FAs
  - ▶ We already know that DFAs and NFAs are equivalent
  - ▶ Now: Equivalence of NFAs and REs

# Transformation of regular expressions into NFAs

- ▶ For a regular expression  $r$ , derive NFA  $\mathcal{A}(r)$  with  $L(\mathcal{A}(r)) = L(r)$ .
- ▶ Idea:
  - ▶ Construct NFAs for the elementary REs ( $\emptyset, \varepsilon, c \in \Sigma$ )
  - ▶ We combine NFAs for subexpressions to generate NFAs for composite REs
- ▶ All NFAs we construct have a number of special properties:
  - ▶ There are no transitions **to the initial state**.
  - ▶ There is only **a single final state**.
  - ▶ There are no transitions **from the final state**.



# Transformation of regular expressions into NFAs

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**We can easily achieve this with  $\varepsilon$ -transitions!**

# Reminder: Regular Expression

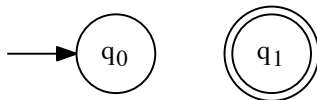
Let  $\Sigma$  be an alphabet.

- ▶ The elementary regular expressions over  $\Sigma$  are:
  - ▶  $\emptyset$  with  $L(\emptyset) = \{\}$
  - ▶  $\varepsilon$  with  $L(\varepsilon) = \{\varepsilon\}$
  - ▶  $c \in \Sigma$  with  $L(c) = \{c\}$
- ▶ Let  $r_1$  and  $r_2$  be regular expressions over  $\Sigma$ .  
Then the following are also regular expressions over  $\Sigma$ :
  - ▶  $r_1 + r_2$  with  $L(r_1 + r_2) = L(r_1) \cup L(r_2)$
  - ▶  $r_1 \cdot r_2$  with  $L(r_1 \cdot r_2) = L(r_1) \cdot L(r_2)$
  - ▶  $r_1^*$  with  $L(r_1^*) = (L(r_1))^*$

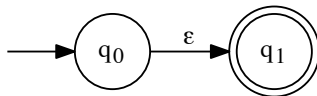
# NFAs for elementary REs

Let  $\Sigma$  be the alphabet which  $r$  is based on.

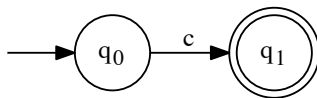
**1**  $\mathcal{A}(\emptyset) = (\{q_0, q_1\}, \Sigma, \{\}, q_0, \{q_1\})$



**2**  $\mathcal{A}(\varepsilon) = (\{q_0, q_1\}, \Sigma, \{(q_0, \varepsilon, q_1)\}, q_0, \{q_1\})$



**3**  $\mathcal{A}(c) = (\{q_0, q_1\}, \Sigma, \{(q_0, c, q_1)\}, q_0, \{q_1\})$  for all  $c \in \Sigma$

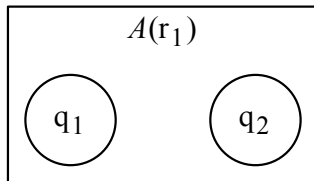


## NFAs for composite REs (general)

- ▶ Assume in the following:
  - ▶  $\mathcal{A}(r_1) = (Q_1, \Sigma, \Delta_1, q_1, \{q_2\})$
  - ▶  $\mathcal{A}(r_2) = (Q_2, \Sigma, \Delta_2, q_3, \{q_4\})$
  - ▶  $Q_1 \cap Q_2 = \{\}$
  - ▶  $q_0, q_5 \notin Q_1 \cup Q_2$
- ▶  $\mathcal{A}(r_1)$  is visualised by a square box with two explicit states
  - ▶ The initial state  $q_1$  is on the left
  - ▶ The unique accepting state  $q_2$  on the right
  - ▶ All other states and transitions are implicit
  - ▶ We mark initial/accepting states only for the composite automaton

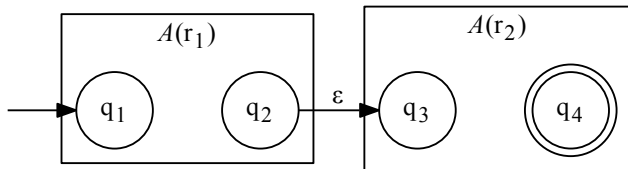
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## NFAs for composite REs (concatenation)

4  $\mathcal{A}(r_1 \cdot r_2) = (Q_1 \cup Q_2, \Sigma, \Delta_1 \cup \Delta_2 \cup \{(q_2, \varepsilon, q_3)\}, q_1, \{q_4\})$

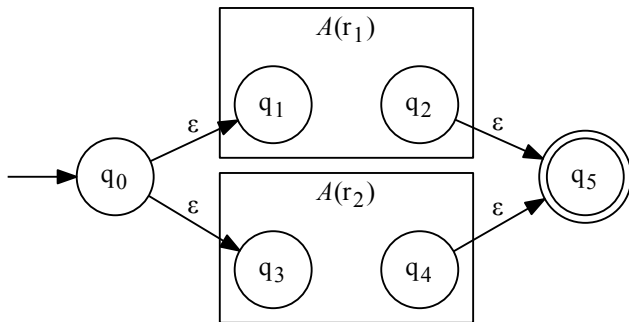


Reminder:

- ▶  $\mathcal{A}(r_1) = (Q_1, \Sigma, \Delta_1, q_1, \{q_2\})$
- ▶  $\mathcal{A}(r_2) = (Q_2, \Sigma, \Delta_2, q_3, \{q_4\})$

## NFAs for composite REs (alternatives)

- 5  $\mathcal{A}(r_1 + r_2) = (\{q_0, q_5\} \cup Q_1 \cup Q_2, \Sigma, \Delta, q_0, \{q_5\})$   
 $\Delta = \Delta_1 \cup \Delta_2 \cup \{(q_0, \varepsilon, q_1), (q_0, \varepsilon, q_3), (q_2, \varepsilon, q_5), (q_4, \varepsilon, q_5)\}$

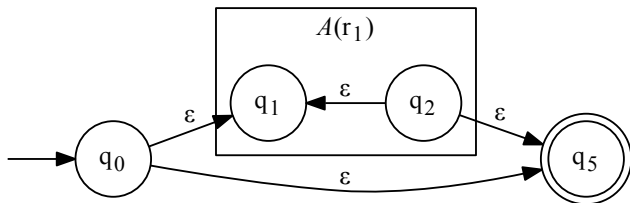


Reminder:

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- ▶  $\mathcal{A}(r_2) = (Q_2, \Sigma, \Delta_2, q_3, \{q_4\})$

## NFAs for composite REs (Kleene Star)

- 6  $\mathcal{A}(r_1^*) = (\{q_0, q_5\} \cup Q_1, \Sigma, \Delta, q_0, \{q_5\})$   
 $\Delta = \Delta_1 \cup \{(q_0, \varepsilon, q_1), (q_2, \varepsilon, q_1), (q_0, \varepsilon, q_5), (q_2, \varepsilon, q_5)\}$



Reminder:

- $\mathcal{A}(r_1) = (Q_1, \Sigma, \Delta_1, q_1, \{q_2\})$



## Result: NFAs can simulate REs

The previous construction produces for each regular expression  $r$  an NFA  $\mathcal{A}$  with  $L(\mathcal{A}) = L(r)$ .

## Result: NFAs can simulate REs

The previous construction produces for each regular expression  $r$  an NFA  $\mathcal{A}$  with  $L(\mathcal{A}) = L(r)$ .

### Corollary

*Every language described by a regular expression can be accepted by a non-deterministic finite automaton.*

## Exercise: transformation of RE into NFA

- ▶ Systematically construct an NFA accepting the same language as the regular expression

$$(a + b)a^*b$$

- ▶ Find arbitrary  $w_1, w_2 \in L((a + b)a^*b)$  with  $|w_1|, |w_2| \geq 5$  and find an accepting run for each

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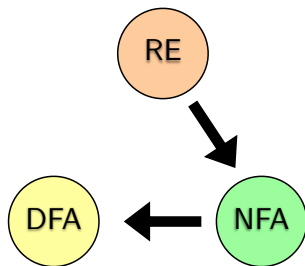
Solution

# Overview and orientation

- ▶ Claim: NFAs, DFAs and REs all describe the **same** language class

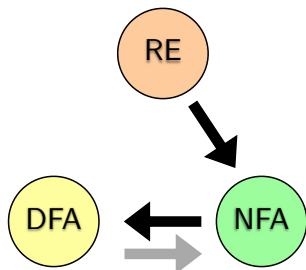
# Overview and orientation

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  - ▶ REs into equivalent NFAs
  - ▶ NFAs into equivalent DFAs



# Overview and orientation

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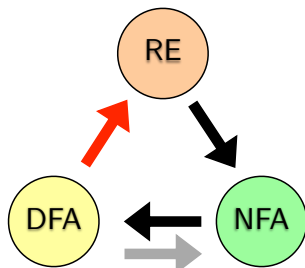




# Overview and orientation

- ▶ Claim: NFAs, DFAs and REs all describe the **same** language class
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**Todo: convert DFA to equivalent RE**



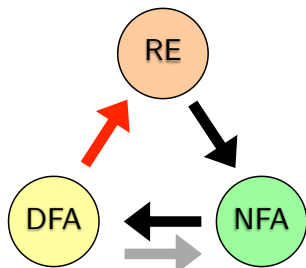
# Overview and orientation

- ▶ Claim: NFAs, DFAs and REs all describe the **same** language class
- ▶ Previous transformations:
  - ▶ REs into equivalent NFAs
  - ▶ NFAs into equivalent DFAs
  - ▶ (DFAs to equivalent NFAs)

**Todo: convert DFA to equivalent RE**

- ▶ Given a DFA  $\mathcal{A}$ , derive a regular expression  $r(\mathcal{A})$  accepting the same language:

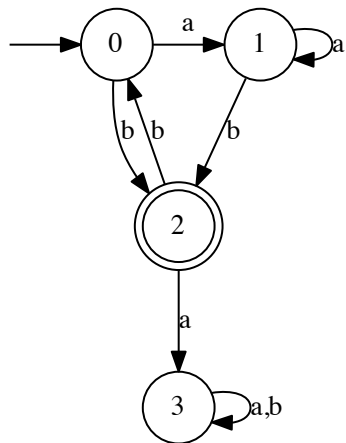
$$L(r(\mathcal{A})) = L(\mathcal{A})$$



# Convert DFA into RE

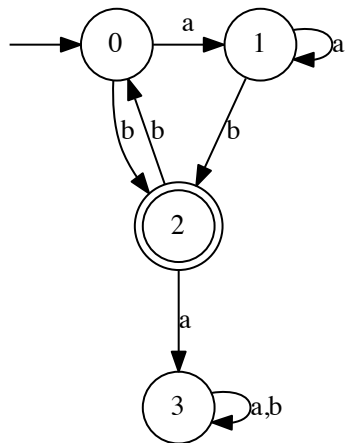
- ▶ Goal: transform DFA  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$  into RE  $r(\mathcal{A})$  with  $L(r(\mathcal{A})) = L(\mathcal{A})$
- ▶ Idea:
  - ▶ For each state  $q$  generate an **equation** describing the language  $L_q$ 
    - ▶ that is accepted when **starting from  $q$** ,
    - ▶ depending on the languages accepted at neighbouring states.
  - ▶ Method:
    - ▶ For each transition with  $c$  to  $q'$ : generate alternative  $c \cdot L_{q'}$
    - ▶ For final states: additionally  $\varepsilon$
- ▶ Solve the resulting system for  $L_{q_0}$ 
  - ▶ Result: RE describing  $L_{q_0} = L(\mathcal{A})$
- ▶ Convention:
  - ▶ States are named  $\{0, 1, \dots, n\}$
  - ▶ Initial state is 0

# Convert DFA to RE: Example



►  $L_0 \doteq$

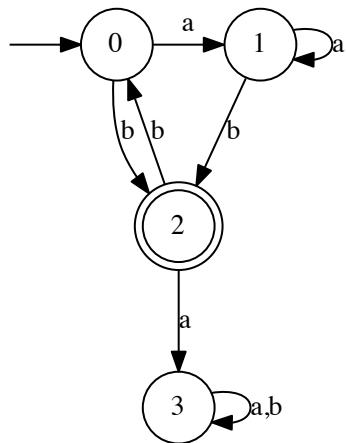
# Convert DFA to RE: Example



▶  $L_0 \doteq aL_1 + bL_2$

▶  $L_1 \doteq$

# Convert DFA to RE: Example

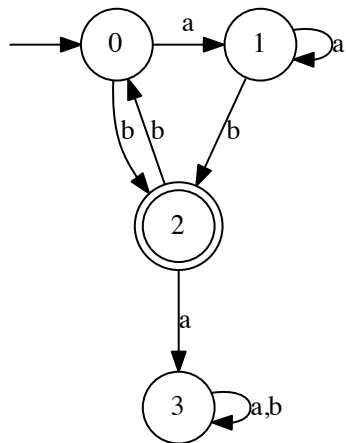


▶  $L_0 \doteq aL_1 + bL_2$

▶  $L_1 \doteq aL_1 + bL_2$

▶  $L_2 \doteq$

# Convert DFA to RE: Example



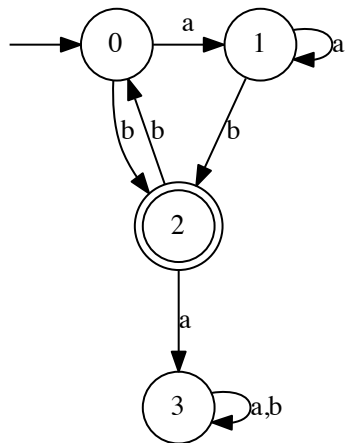
▶  $L_0 \doteq aL_1 + bL_2$

▶  $L_1 \doteq aL_1 + bL_2$

▶  $L_2 \doteq aL_3 + bL_0 + \varepsilon$

▶  $L_3 \doteq$

# Convert DFA to RE: Example



▶  $L_0 \doteq aL_1 + bL_2$

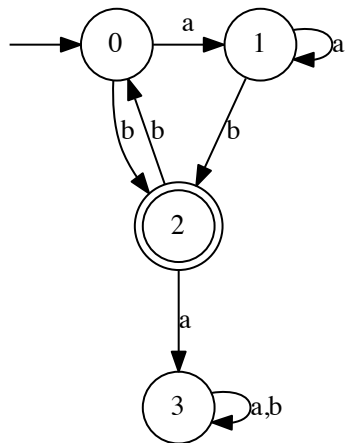
▶  $L_1 \doteq aL_1 + bL_2$

▶  $L_2 \doteq aL_3 + bL_0 + \varepsilon$

▶  $L_3 \doteq (a + b)L_3$



# Convert DFA to RE: Example



▶  $L_0 \doteq aL_1 + bL_2$

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▶  $L_2 \doteq aL_3 + bL_0 + \varepsilon$

▶  $L_3 \doteq (a + b)L_3$

4 equations, 4 unknowns

What now?

# Insert: Arden's Lemma

Lemma:

$$\varepsilon \notin L(s) \text{ and } r \doteq sr + t \longrightarrow r \doteq s^*t$$

Arden, Dean N.:  
*Delayed-logic  
and finite-state  
machines*,  
Proceedings of  
the Second  
Annual  
Symposium on  
Switching  
Circuit Theory  
and Logical  
Design, 1961,  
pp. 133–151,  
IEEE

# Insert: Arden's Lemma

Lemma:

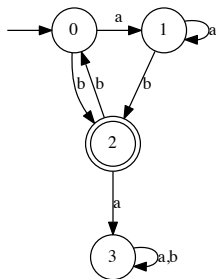
$$\varepsilon \notin L(s) \text{ and } r \doteq sr + t \longrightarrow r \doteq s^*t$$

Compare Arto Salomaa:

$$\varepsilon \notin L(s) \text{ and } r \doteq rs + t \longrightarrow r \doteq ts^*$$

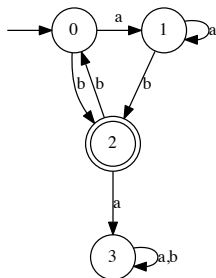
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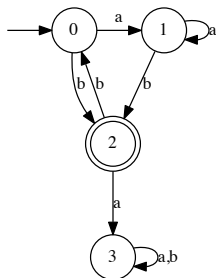
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$$L_3 \doteq (a + b)L_3 + \emptyset \quad \text{[neutral el.]}$$

# Convert DFA to RE: Example



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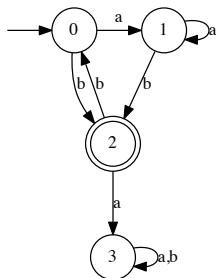
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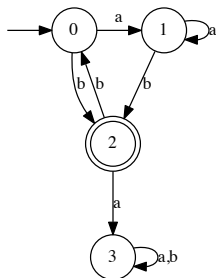
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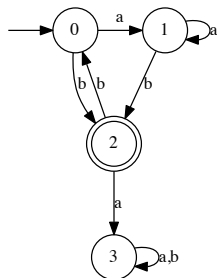


- ▶  $L_0 \doteq aL_1 + bL_2$
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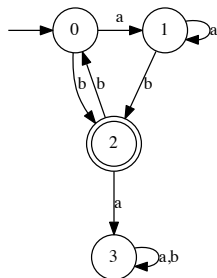
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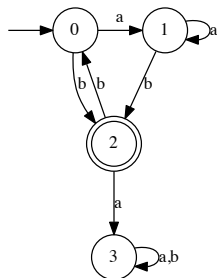
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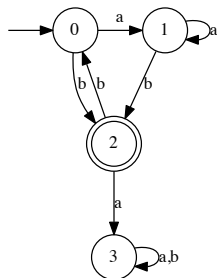
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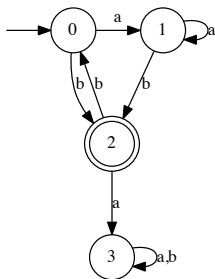
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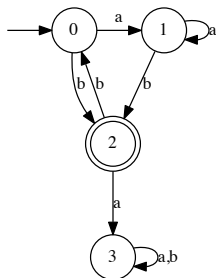
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# Convert DFA to RE: Example (continued)



- ▶  $L_0 \doteq aL_1 + bL_2$
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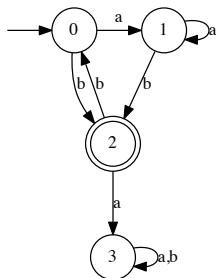
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$$L_0 \doteq a(a^*b(bL_0 + \varepsilon)) + b(bL_0 + \varepsilon) \quad \text{[replace } L_1, L_2]$$

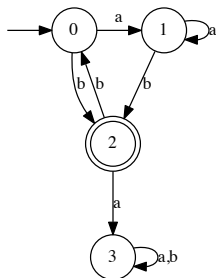
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$$\begin{aligned} L_0 &\doteq a(a^*b(bL_0 + \varepsilon)) + b(bL_0 + \varepsilon) && \text{[replace } L_1, L_2\text{]} \\ &\doteq aa^*bbL_0 + aa^*b + bbL_0 + b && \text{[dist.]} \end{aligned}$$

# Convert DFA to RE: Example (continued)

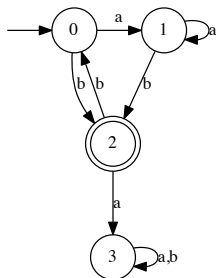


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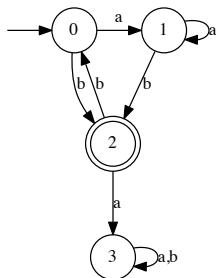
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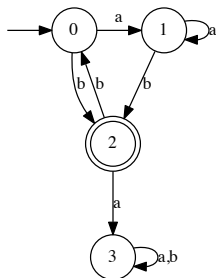
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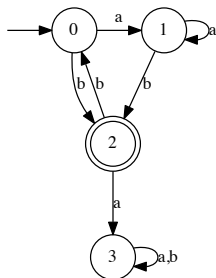
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# Convert DFA to RE: Example (continued)



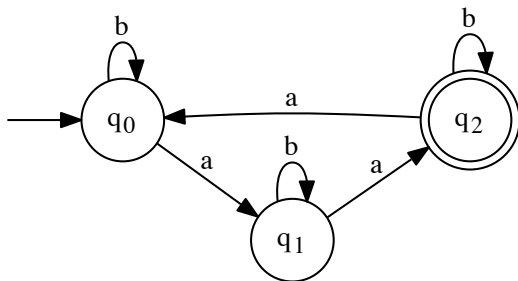
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 &\doteq (aa^*bb + bb)^*(aa^*b + b) && \text{[Arden]} \\
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 &\doteq (a^*bb)^*(a^*b) && \text{[ } rr^* + \varepsilon \doteq r^*\text{]}
 \end{aligned}$$

Therefore:  $L(\mathcal{A}) = L((a^*bb)^*(a^*b))$

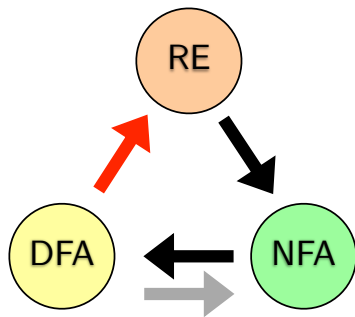
## Exercise: conversion from DFA to RE

Transform the following DFA into a regular expression accepting the same language:



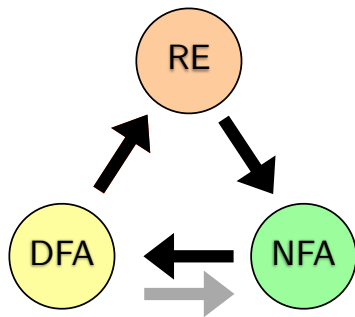
# Resume: Finite automata and regular expressions

- ▶ We have learned how to convert
  - ▶ REs to equivalent NFAs
  - ▶ NFAs to equivalent DFAs
  - ▶ (DFAs to equivalent NFAs)



# Resume: Finite automata and regular expressions

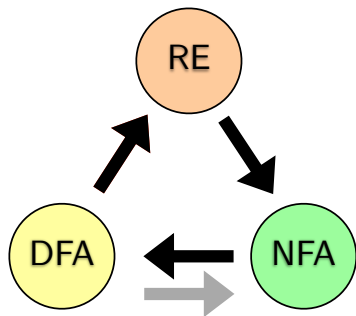
- ▶ We have learned how to convert
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  - ▶ (DFAs to equivalent NFAs)
  - ▶ **DFAs to equivalent REs**



# Resume: Finite automata and regular expressions

- ▶ We have learned how to convert
  - ▶ REs to equivalent NFAs
  - ▶ NFAs to equivalent DFAs
  - ▶ (DFAs to equivalent NFAs)
  - ▶ **DFAs to equivalent REs**

**REs, NFAs and DFAs describe the same class of languages – regular languages!**





**and now it's time for something  
completely different**



**and now it's time for something  
completely different**



End lecture 6

Introduction

Regular Languages and Finite Automata

Regular Expressions

**Finite Automata**

Non-Determinism

Regular expressions and Finite Automata

**Minimisation**

Equivalence

The Pumping Lemma

Properties of Regular Languages

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

# Efficient Automata: Minimisation of DFAs

Given the DFA

$$\mathcal{A} = (Q, \Sigma, \delta, q_0, F),$$

we want to derive a DFA

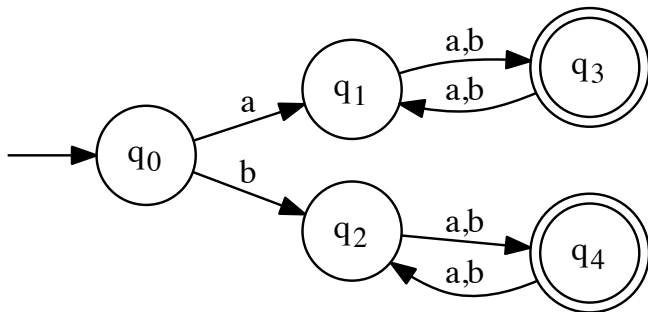
$$\mathcal{A}^- = (Q^-, \Sigma, \delta^-, q_0, F^-),$$

accepting the same language:

$$L(\mathcal{A}) = L(\mathcal{A}^-)$$

for which the **number of states** (elements of  $Q^-$ ) is **minimal**, i.e. there is no DFA accepting  $L(\mathcal{A})$  with fewer states.

## Minimisation of DFAs: example/exercise



How small can we make it?

# Minimisation of DFAs

## Basic idea:

- ▶ Two states with both
    - ▶ ... the same acceptance status (accepting/not accepting)
    - ▶ ... exactly the same transitions
- are **equivalent** and can be **merged**

# Minimisation of DFAs

## Basic idea:

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**Realisation:** For a DFA  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ , identify **pairs of necessarily distinct states**

- ▶ Base case: Two states  $p, q$  are necessarily distinct if:
  - ▶ one of them is accepting, the other is not accepting
- ▶ Inductive case: Two states  $p, q$  are necessarily distinct if
  - ▶ there is a  $c \in \Sigma$  such that  $\delta(p, c) = p', \delta(q, c) = q'$
  - ▶ and  $p', q'$  are already necessarily distinct



# Identification of distinct/equivalent states

## Definition (Necessarily distinct states)

For a DFA  $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$ ,  $V$  is the smallest set of pairs with

- ▶  $\{(p, q) \in (Q \times Q) \mid p \in F, q \notin F\} \subseteq V$
  - ▶  $\{(p, q) \in (Q \times Q) \mid p \notin F, q \in F\} \subseteq V$
  - ▶ if  $\delta(p, c) = p', \delta(q, c) = q', (p', q') \in V$  for some  $c \in \Sigma$ , then  $(p, q) \in V$ .
- 
- ▶ We identify pairs of necessarily distinct states
  - ▶ Pairs of states that are not necessarily distinct are equivalent and can be merged

# Minimisation of DFAs

- 1 Initialize  $V$  with all those pairs for which one member is a final state and the other is not:

$$V = \{(p, q) \in Q \times Q \mid (p \in F \wedge q \notin F) \vee (p \notin F \wedge q \in F)\}.$$

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- 2 While there exists
  - ▶ a new pair of states  $(p, q)$  and a symbol  $c$
  - ▶ such that the states  $\delta(p, c)$  and  $\delta(q, c)$  are necessarily distinct,
  - ▶ add this pair and its inverse to  $V$ :

# Minimisation of DFAs

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```
while ( $\exists(p, q) \in Q \times Q \exists c \in \Sigma \mid (\delta(p, c), \delta(q, c)) \in V \wedge (p, q) \notin V$ )
{
   $V = V \cup \{(p, q), (q, p)\}$ 
}
```

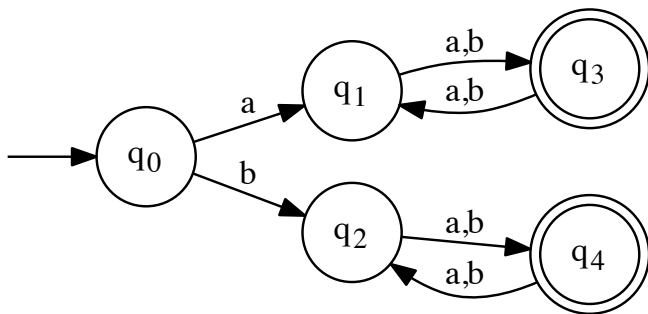
# Minimisation of DFAs: merging States

- ▶ If there is a pair of states  $(p, q)$  such that for every word  $w \in \Sigma^*$ 
  - ▶ reading  $w$  results in equivalent successor states,
  - ▶ then  $p$  and  $q$  are equivalent.

$(p, q) \notin V \Rightarrow \forall w \in \Sigma^* : \delta(p, w) \text{ and } \delta(q, w) \text{ are indistinguishable.}$
- ▶ Equivalent states  $p, q$  can be **merged**
  - ▶ Replace all transitions to  $p$  by transitions to  $q$
  - ▶ Remove  $p$
- ▶ This process can be iterated to identify and merge all pairs of equivalent states

# Minimisation of DFAs: example

We want to minimize this DFA with 5 states:



## Minimisation of DFAs: example (cont.)

This is the formal definition of the DFA:

$$\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$$

with

1  $Q = \{q_0, q_1, q_2, q_3, q_4\}$

2  $\Sigma = \{a, b\}$

3  $\delta = \dots$  (skipped to save space, see graph)

4  $F = \{q_3, q_4\}$

## Minimisation of DFAs: example (cont.)

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Represent the set  $V$  by means of a two-dimensional table with

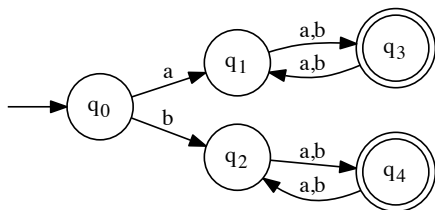
- ▶ the elements of  $Q$  as columns and rows
- ▶ the elements of  $V$  are marked with  $\times$
- ▶ pairs that are definitely **not** members of  $V$  are marked with  $\circ$



# Minimisation of DFAs: example (cont.)

- 1** the initial state of  $V$  is obtained by using  $F = \{q_3, q_4\}$  and  $Q \setminus F = \{q_0, q_1, q_2\}$ :

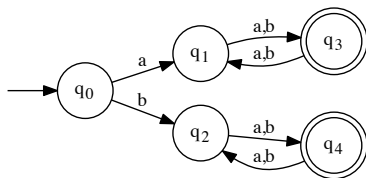
	$q_0$	$q_1$	$q_2$	$q_3$	$q_4$
$q_0$				×	×
$q_1$				×	×
$q_2$				×	×
$q_3$	×	×	×		
$q_4$	×	×	×		



## Minimisation of DFAs: example (cont.)

- 2 The elements of  $\{(q_i, q_i) \mid i \in \{0, \dots, 4\}\}$  are not contained in  $V$  since every state is equivalent to itself:

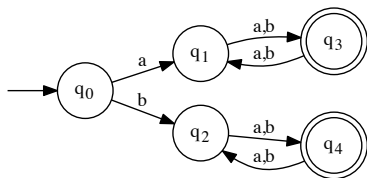
	$q_0$	$q_1$	$q_2$	$q_3$	$q_4$
$q_0$	○			×	×
$q_1$		○		×	×
$q_2$			○	×	×
$q_3$	×	×	×	○	
$q_4$	×	×	×		○



## Minimisation of DFAs: example (cont.)

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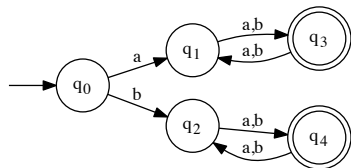


There are eight remaining empty fields. Since the table is symmetric, **four** pairs of states have to be checked.

## Minimisation of DFAs: example (cont.)

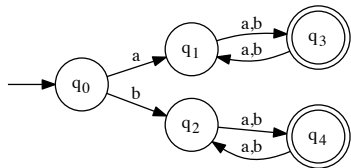
## Minimisation of DFAs: example (cont.)

- 3 Check the transitions of **every remaining state-pair** for **every letter**.



# Minimisation of DFAs: example (cont.)

- 3** Check the transitions of **every remaining state-pair** for **every letter**.



- 1**  $\delta(q_0, a) = q_1; \delta(q_1, a) = q_3; (q_1, q_3) \in V \rightarrow (q_0, q_1), (q_1, q_0) \in V$   
**2**  $\delta(q_0, a) = q_1; \delta(q_2, a) = q_4; (q_1, q_4) \in V \rightarrow (q_0, q_2), (q_2, q_0) \in V$   
**3**  $\delta(q_1, a) = q_3; \delta(q_2, a) = q_4; (q_3, q_4) \notin V$  (as of yet)  
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 $\delta(q_3, b) = q_1; \delta(q_4, b) = q_2; (q_1, q_2) \notin V$  (as of yet)

## Minimisation of DFAs: example (cont.)

- 4 Mark the newly found distinct pairs with  $\times$ :

	$q_0$	$q_1$	$q_2$	$q_3$	$q_4$
$q_0$	○	×	×	×	×
$q_1$	×	○		×	×
$q_2$	×		○	×	×
$q_3$	×	×	×	○	
$q_4$	×	×	×		○

## Minimisation of DFAs: example (cont.)

- 4 Mark the newly found distinct pairs with  $\times$ :

	$q_0$	$q_1$	$q_2$	$q_3$	$q_4$
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$q_2$	×		○	×	×
$q_3$	×	×	×	○	
$q_4$	×	×	×		○

Two pairs remain to be checked.



## Minimisation of DFAs: example (cont.)

- 5 Check the remaining pairs.
- 6 Since no additional necessarily distinct state pairs are found, fill empty cells with  $\circ$ :

	$q_0$	$q_1$	$q_2$	$q_3$	$q_4$
$q_0$	$\circ$	$\times$	$\times$	$\times$	$\times$
$q_1$	$\times$	$\circ$	$\circ$	$\times$	$\times$
$q_2$	$\times$	$\circ$	$\circ$	$\times$	$\times$
$q_3$	$\times$	$\times$	$\times$	$\circ$	$\circ$
$q_4$	$\times$	$\times$	$\times$	$\circ$	$\circ$

## Minimisation of DFAs: example (cont.)

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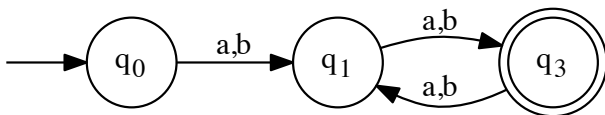
	$q_0$	$q_1$	$q_2$	$q_3$	$q_4$
$q_0$	$\circ$	$\times$	$\times$	$\times$	$\times$
$q_1$	$\times$	$\circ$	$\circ$	$\times$	$\times$
$q_2$	$\times$	$\circ$	$\circ$	$\times$	$\times$
$q_3$	$\times$	$\times$	$\times$	$\circ$	$\circ$
$q_4$	$\times$	$\times$	$\times$	$\circ$	$\circ$

From the table, we can derive the following pairs of equivalent states (omitting trivial and symmetric ones):

- ▶  $(q_1, q_2)$ ,
- ▶  $(q_3, q_4)$ .

## Minimisation of DFAs: example (cont.)

- ▶ This is the minimized DFA after merging equivalent states:



# Minimisation of DFAs: exercise

Derive a minimal DFA accepting the language

$$L(a(ba)^*).$$

Solve the exercise in three steps:

- 1 Derive an NFA accepting  $L$ .
- 2 Transform the NFA into a DFA.
- 3 Minimize the DFA.

# Uniqueness of minimal DFA

## Theorem (The minimal DFA is unique)

*Assume an arbitrary regular language  $L$ . Then there is a unique (up to the the renaming of states) **minimal** DFA  $\mathcal{A}$  with  $L(\mathcal{A}) = L$ .*

- ▶ States can easily be systematically renamed to make equivalent minimal automata strictly equal
- ▶ The unique minimal DFA for  $L$  can be constructed by minimizing an arbitrary DFA that accepts  $L$

Introduction

Regular Languages and Finite Automata

Regular Expressions

**Finite Automata**

Non-Determinism

Regular expressions and Finite Automata

Minimisation

**Equivalence**

The Pumping Lemma

Properties of Regular Languages

Scanners and Flex

Formal Grammars and Context-Free Languages

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Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

# Equivalence of regular expressions

- ▶ Different regular expressions can describe the **same language**
- ▶ **Algebraic transformation rules** can be used to prove equivalence
  - ▶ requires human interaction
  - ▶ can be very difficult
  - ▶ non-equivalence cannot be shown
- ▶ Now: straight-forward algorithm proving equivalence of REs based on FA
- ▶ The algorithm is described in the textbook by John E. Hopcroft, Rajeev Motwani, Jeffrey D. Ullman: *Introduction to Automata Theory, Languages, and Computation (3rd edition)*, 2007 (and earlier editions)

# Equivalence of regular expressions: algorithm

- 1 Given the REs  $r_1$  and  $r_2$ , derive NFAs  $\mathcal{A}_1$  and  $\mathcal{A}_2$  accepting their respective languages:

$$L(r_1) = L(\mathcal{A}_1) \quad \text{and} \quad L(r_2) = L(\mathcal{A}_2).$$

- 2 Transform the NFAs  $\mathcal{A}_1$  and  $\mathcal{A}_2$  into the DFAs  $\mathcal{D}_1$  and  $\mathcal{D}_2$ .
- 3 Minimize the DFAs  $\mathcal{D}_1$  and  $\mathcal{D}_2$  yielding the DFAs  $\mathcal{M}_1$  and  $\mathcal{M}_2$ .
- 4  $r_1 \doteq r_2$  holds iff  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are identical (modulo renaming of states)



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Note: If equivalence can be shown in any intermediate stage of the algorithm, this is sufficient to prove  $r_1 \doteq r_2$  (e.g. if  $\mathcal{A}_1 = \mathcal{A}_2$ ).

## Exercise: Equivalence of regular expressions

Reusing an exercise from an earlier section, prove the following equivalence (by conversion to minimal DFAs):

$$10(10)^* \doteq 1(01)^*0$$

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Solution

End lecture 7

# Outline

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# Non-regular languages

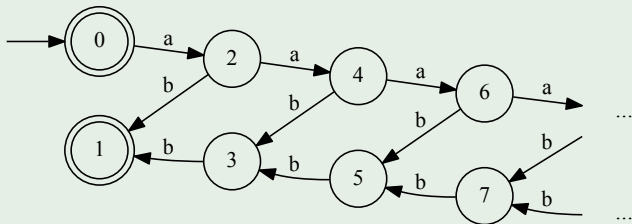
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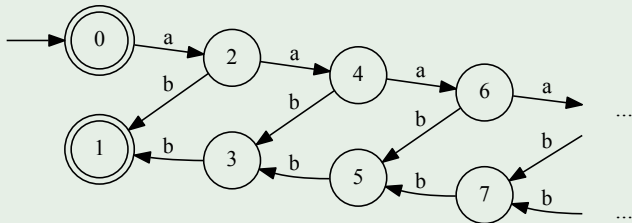


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- ▶ Is there a better solution?
- ▶ If no, how can this be shown?

# Pumping Lemma: Idea

- 1 Every regular language  $L$  is accepted by a **finite** Automaton  $\mathcal{A}_L$ .
- 2 If  $L$  contains arbitrarily long words, then  $\mathcal{A}_L$  must contain a **cycle**.
  - ▶  $L$  contains arbitrarily long words iff  $L$  is infinite.
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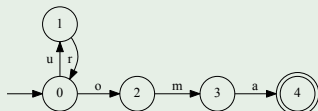


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## Example (Cyclic Automaton $\mathcal{C}$ )

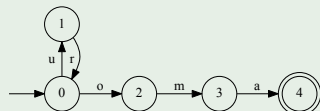


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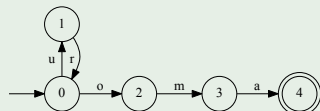
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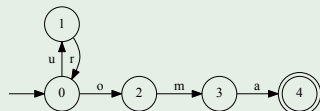
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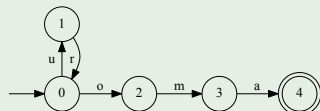
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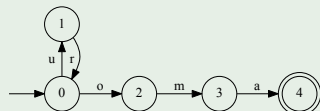
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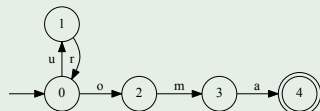
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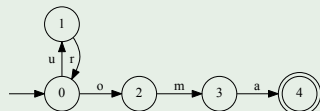


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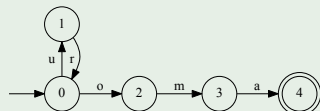
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# The Pumping Lemma

## Lemma

*Let  $L$  be a regular language.*

*Then there exists a  $k \in \mathbb{N}$  such that for every word  $s \in L$  with  $|s| \geq k$  the following holds:*

# The Pumping Lemma

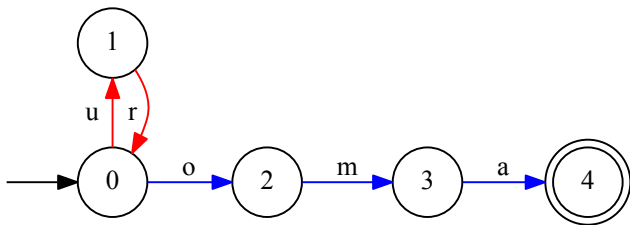
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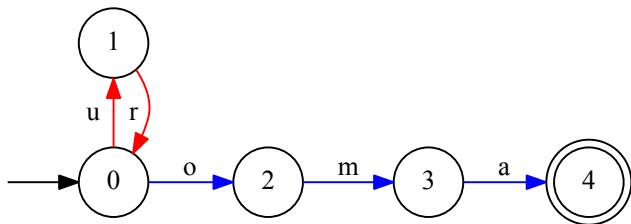
Then there exists a  $k \in \mathbb{N}$  such that for every word  $s \in L$  with  $|s| \geq k$  the following holds:

- 1  $\exists u, v, w \in \Sigma^* (s = u \cdot v \cdot w)$ ,  
i.e.  $s$  consists of *prolog*  $u$ , *cycle*  $v$  and *epilog*  $w$ ,
- 2  $v \neq \varepsilon$ ,  
i.e. the cycle has a *length of at least 1*,
- 3  $|u \cdot v| \leq k$ ,  
i.e. prolog and cycle combined have a *length of at most  $k$* ,
- 4  $\forall h \in \mathbb{N} (u \cdot v^h \cdot w \in L)$ ,  
i.e. an *arbitrary number of cycle transitions* results in a word of the language  $L$ .

# The Pumping Lemma visualised



# The Pumping Lemma visualised



- ▶  $\mathcal{C}$  has 5 states  $k = 5$
- ▶ `uroma` has 5 letters  $s = \text{uroma}$
- ▶ There is a segmentation  $s = u \cdot v \cdot w$   $u = \varepsilon$   $v = \text{ur}$   $w = \text{oma}$
- ▶ such that  $v \neq \varepsilon$   $v = \text{ur}$
- ▶ and  $|u \cdot v| \leq k$   $|\varepsilon \cdot \text{ur}| = 2 \leq 5$
- ▶ and  $\forall h \in \mathbb{N}(u \cdot v^h \cdot w \in L(\mathcal{C}))$   $(\text{ur})^* \text{oma} \subseteq L(\mathcal{C})$

# Using the Pumping Lemma

- ▶ The Pumping Lemma describes a property of **regular** languages
  - ▶ *If  $L$  is regular, then some words can be pumped up.*
- ▶ Goal: proof of **irregularity** of a language
  - ▶ *If  $L$  has property  $X$ , then  $L$  is not regular.*
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## Theorem (Contraposition)

$$A \rightarrow B \quad \Leftrightarrow \quad \neg B \rightarrow \neg A$$



# Contraposition of the Pumping Lemma

The Pumping Lemma in formal logic:

$$\begin{aligned} \text{reg}(L) \rightarrow & \exists k \in \mathbb{N} \forall s \in L : (|s| \geq k \rightarrow \\ & \exists u, v, w : (s = u \cdot v \cdot w \wedge v \neq \varepsilon \wedge |u \cdot v| \leq k \wedge \\ & \forall h \in \mathbb{N} : (u \cdot v^h \cdot w \in L))) \end{aligned}$$

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Contraposition of the PL:

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After pushing negation inward and doing some propositional transformations:

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If for each natural number  $k$  there is a word  $s$  with length at least  $k$  and for every segmentation  $u \cdot v \cdot w$  of  $s$



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then  $L$  is not regular.

# Proving Irregularity for a Language

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## Example ( $L = a^n b^n$ )

- ▶ Choose  $s = a^k b^k$ . It follows:

$$s = \underbrace{a^i}_u \cdot \underbrace{a^j}_v \cdot \underbrace{a^\ell \cdot b^k}_w$$

- ▶  $i + j + \ell = k$
- ▶ since  $|u \cdot v| \leq k$  holds,  $u$  and  $v$  consist only of  $a$ 's
- ▶  $v \neq \varepsilon$  implies  $j \geq 1$
- ▶ Choose  $h = 0$ . It follows:
  - ▶  $u \cdot v^h \cdot w = u \cdot w = a^{i+\ell} b^k$
  - ▶  $j \geq 1$  implies  $i + \ell < k$
  - ▶  $a^{i+\ell} b^k \notin L$



# Regarding quantifiers

Four quantifiers:

- ▶ In the lemma:

$$\exists k \forall s \exists u, v, w \forall h (u \cdot v^h \cdot w \in L)$$

- ▶ To show irregularity:

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To do:

- 1 Find a word  $s$  depending on the length  $k$ .
- 2 Find an  $h$  depending on the segmentation  $u \cdot v \cdot w$ .
- 3 Prove that  $u \cdot v^h \cdot w \notin L$  holds.

## Exercise: The Pumping Game

Play the pumping game at <http://weitz.de/pump/>.

Remark: *Weitz uses a slightly stronger variant of the pumping lemma, where the string  $uv$  can be anywhere in the word, not just at the beginning.*

## Exercise: $a^n b^m$ with $n < m$

Use the pumping lemma to show that

$$L = \{a^n b^m \mid n < m\}$$

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Solution

## Challenging exercise / homework

Let  $L$  be the number containing all words of the form  $a^p$  where  $p$  is a prime number:

$$L = \{a^p \mid p \in \mathbb{P}\}.$$

Prove that  $L$  is not a regular language.

Hint: let  $h = p + 1$

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Finite automata cannot **count** arbitrarily high.

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## Examples (Nested dependencies)

**C** for every { there is a }

**XML** for every <token> there is a </token>

**L<sup>A</sup>T<sub>E</sub>X** for every \begin{env} there is a \end{env}

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**German** for every subject there is a predicate

```
Erinnern Sie sich,  
    wie der Krieger,  
        der die Botschaft,  
            die den Sieg,  
                den die Griechen bei Marathon  
                    errungen hatten,  
                        verkündete,  
                            brachte,  
                                starb!
```

# Pumping Lemma: Summary

- ▶ Every regular language is accepted by a DFA  $\mathcal{A}$  (with  $k$  states).
- ▶ Pumping lemma: words with at least  $k$  letters can be **pumped up**.
- ▶ If it is possible to pump up a word  $w \in L$  and obtain a word  $w' \notin L$ , then  $L$  is **not regular**.
  - ▶ Make sure to handle quantifiers correctly!
- ▶ Practical relevance
  - ▶ FAs cannot **count arbitrarily high**.
  - ▶ **Nested structures** are not regular.
    - ▶ programming languages
    - ▶ natural languages
  - ▶ More powerful tools are needed to handle these languages.

# Outline

Introduction

**Regular Languages and Finite Automata**

Regular Expressions

Finite Automata

The Pumping Lemma

**Properties of Regular Languages**

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

# Regular languages: Closure properties

Reminder:

- ▶ **Formal languages** are sets of words (over a finite alphabet)
- ▶ A formal language  $L$  is a *regular language* if any of the following holds:
  - ▶ There exists an NFA  $\mathcal{A}$  with  $L(\mathcal{A}) = L$
  - ▶ There exists a DFA  $\mathcal{A}$  with  $L(\mathcal{A}) = L$
  - ▶ There exists a regular expression  $r$  with  $L(r) = L$
  - ▶ There exists a regular *grammar*  $G$  with  $L(G) = L$
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## Question

**What can we do to regular languages and be sure the result is still regular?**

## Closure properties (Question)

Question: If  $L_1$  and  $L_2$  are regular languages, does the same hold for  $L_1 \cup L_2$ ? (closure under **union**)



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**Meta-Question: How do we answer these questions?**

# Closure properties (Theorem)

## Theorem (Closure properties of regular languages)

*Let  $L_1$  and  $L_2$  be regular languages. Then the following languages are also regular:*

- ▶  $L_1 \cup L_2$
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## Proof.

Idea: using (disjoint) finite automata for  $L_1$  and  $L_2$ , construct an automaton for the different languages above. □



# Closure under union, concatenation, and Kleene-star

We use the same construction that was used to generate NFAs for regular expressions:

Let  $\mathcal{A}_{L_1}$  and  $\mathcal{A}_{L_2}$  be automata for  $L_1$  and  $L_2$ .

$L_1 \cup L_2$  new initial and final states,

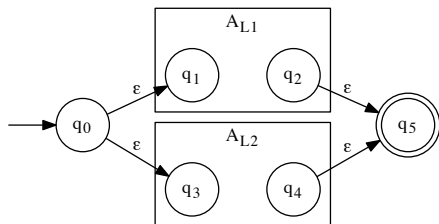
$\varepsilon$ -transitions to initial/final states of  $\mathcal{A}_{L_1}$  and  $\mathcal{A}_{L_2}$

$L_1 \cdot L_2$   $\varepsilon$ -transition from final state of  $\mathcal{A}_{L_1}$  to initial state of  $\mathcal{A}_{L_2}$

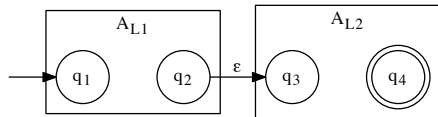
- $(L_1)^*$
- ▶ new initial and final states (with  $\varepsilon$ -transitions),
  - ▶  $\varepsilon$ -transitions from the original final states to the original initial state,
  - ▶  $\varepsilon$ -transition from the new initial to the new final state.

# Visual refresher

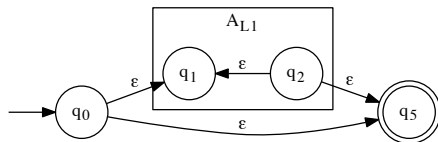
$L_1 \cup L_2$



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# Closure under intersection

Let  $\mathcal{A}_{L_1} = (Q_1, \Sigma, \delta_1, q_{0_1}, F_1)$  and  $\mathcal{A}_{L_2} = (Q_2, \Sigma, \delta_2, q_{0_2}, F_2)$  be DFAs for  $L_1$  and  $L_2$ .

An automaton  $L = (Q, \Sigma, \delta, q_0, F)$  for  $\mathcal{A}_{L_1} \cap \mathcal{A}_{L_2}$  can be generated as follows:

- ▶  $Q = Q_1 \times Q_2$
- ▶  $\delta((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$  for all  $q_1 \in Q_1, q_2 \in Q_2, a \in \Sigma$
- ▶  $q_0 = (q_{0_1}, q_{0_2})$
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- ▶  $q_0 = (q_{0_1}, q_{0_2})$
- ▶  $F = F_1 \times F_2$

This **product automaton**

- ▶ starts in state that corresponds to initial states of  $\mathcal{A}_{L_1}$  and  $\mathcal{A}_{L_2}$ ,
- ▶ simulates simultaneous processing in both automata
- ▶ accepts if both  $\mathcal{A}_{L_1}$  and  $\mathcal{A}_{L_2}$  accept.

## Exercise: Product automaton

Generate automata for

- ▶  $L_1 = \{w \in \{0, 1\}^* \mid |w|_1 \text{ is divisible by } 2\}$
- ▶  $L_2 = \{w \in \{0, 1\}^* \mid |w|_1 \text{ is divisible by } 3\}$

Then generate an automaton for  $L_1 \cap L_2$ .

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Solution

End lecture 8

# Closure under complement

## Theorem (Closure under complement)

*Let  $L$  be a regular language over  $\Sigma$ . Then  $\bar{L} = \Sigma^* \setminus L$  is regular.*

Let  $\mathcal{A}_L = (Q, \Sigma, q_0, \delta, F)$  be a DFA for the language  $L$ .

Then  $\overline{\mathcal{A}_L} = (Q, \Sigma, q_0, \delta, Q \setminus F)$  is an automaton accepting  $\bar{L}$ :



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### Reminder:

$$\delta' : Q \times \Sigma^* \rightarrow Q$$

$\delta'(q_0, w)$  is the final state of the automaton after processing  $w$

# Closure under complement

## Theorem (Closure under complement)

Let  $L$  be a regular language over  $\Sigma$ . Then  $\bar{L} = \Sigma^* \setminus L$  is regular.

Let  $\mathcal{A}_L = (Q, \Sigma, q_0, \delta, F)$  be a DFA for the language  $L$ .

Then  $\bar{\mathcal{A}}_L = (Q, \Sigma, q_0, \delta, Q \setminus F)$  is an automaton accepting  $\bar{L}$ :

- ▶ if  $w \in L(\mathcal{A})$  then  $\delta'(q_0, w) \in F$ , i.e.  
 $\delta'(q_0, w) \notin Q \setminus F$ , which implies  $w \notin L(\bar{\mathcal{A}}_L)$ .
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**All we have to do is exchange accepting and non-accepting states.**

## Closure properties: exercise

Show that  $L = \{w \in \{a, b\}^* \mid |w|_a = |w|_b\}$  is not regular.

Hint: Use the following:

- ▶  $a^n b^n$  is not regular. (Pumping lemma)
- ▶  $a^* b^*$  is regular. (Regular expression)
- ▶ (one of) the closure properties shown before.

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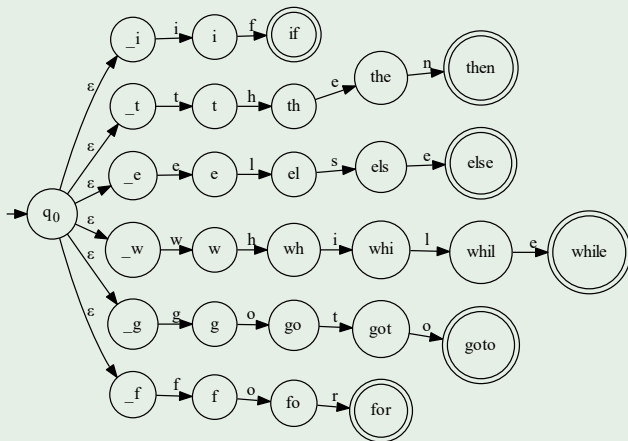
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- ▶ Let  $q_0$  be a new state, from which there is an  $\varepsilon$ -transition to each  $q_{0_i}$ .

Then the resulting automaton, with  $q_0$  as initial state and all  $q_{f_i}$  as final states, accepts  $L$ . □

# Example: finite language

Example ( $L = \{if, then, else, while, goto, for\}$  over  $\Sigma_{ASCII}$ )





# Finite languages and regular expressions

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## Alternate proof.

Let  $L = \{w_1, w_2, \dots, w_n\}$ .

Write  $L$  as the regular expression  $w_1 + w_2 + \dots + w_n$ . □

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## Corollary

*The class of finite languages is characterised by*

- ▶ *acyclic NFAs (or DFAs that have no cycles on any path from the initial state to an accepting state)*
- ▶ *regular expressions without Kleene star.*

# Decision problems

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|------------------------------|---------------------|
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| Is $w$ an element of $L_1$ ? | word problem        |
| Is $L_1$ equal to $L_2$ ?    | equivalence problem |
| Is $L_1$ finite?             | finiteness problem  |

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Algorithm: Let  $\mathcal{A}$  be an automaton accepting the language  $L$ .

- ▶ Starting with the initial state  $q_0$ , mark all states to which there is a transition from  $q_0$  as **reachable**.
- ▶ Continue with transitions from states which are already marked as **reachable** until either a final state is reached or no further states are reachable.
- ▶ If a final (accepting) state is **reachable**, then  $L \neq \{\}$  holds.





## Group exercise: Emptiness problem

- ▶ Find an alternative algorithm for checking emptiness, using the results from the chapter on equivalence.

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Algorithm:

- ▶  $q_1 := \delta(q_0, c_1)$
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All we have to do is simulate the run of  $\mathcal{A}$  on  $w$ .

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## Alternative proof.

One can also use closure properties and decidability of the emptiness problem:

$$L_1 = L_2 \text{ iff } \underbrace{(L_1 \cap \overline{L_2})}_{\text{words that are in } L_1, \text{ but not in } L_2} \cup \underbrace{(\overline{L_1} \cap L_2)}_{\text{words that are not in } L_1, \text{ but in } L_2} = \{\}$$



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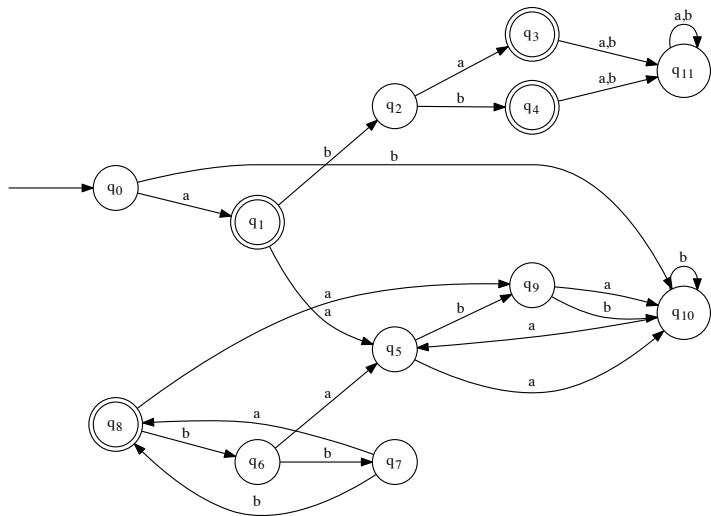
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Note that  $\mathcal{A}_f$  may be a NFA (because it misses some transitions)

# Exercise: Finiteness

Consider the following DFA  $\mathcal{A}$ . Use the previous algorithm to decide if  $L(\mathcal{A})$  is finite. Describe  $L(\mathcal{A})$ .



# Regular languages: summary

## Regular languages

- ▶ are characterised by
  - ▶ NFAs / DFAs
  - ▶ regular expressions
  - ▶ regular grammars
- ▶ can be transferred from one formalism to another one
- ▶ are **closed** under all operators (considered here)
- ▶ all decision problems (considered here) are **decidable**
- ▶ do not contain several interesting languages ( $a^n b^n$ , **counting**)
  - ▶ see chapter on **grammars**
- ▶ can express important features of programming languages
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# Outline

Introduction

Regular Languages and Finite Automata

**Scanners and Flex**

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

# Computing Environment

- ▶ For practical exercises, you will need a complete Linux/UNIX environment. If you do not run one natively, there are several options:
  - ▶ You can install VirtualBox (<https://www.virtualbox.org>) and then install e.g. Ubuntu (<http://www.ubuntu.com/>) on a virtual machine. Make sure to install the *Guest Additions*
  - ▶ For Windows, you can install the **complete** UNIX emulation package Cygwin from <http://cygwin.com>
  - ▶ For MacOS, you can install `fink` (<http://fink.sourceforge.net/>) or MacPorts (<https://www.macports.org/>) and the necessary tools
- ▶ You will need at least `flex`, `bison`, `gcc`, `make`, and a good text editor



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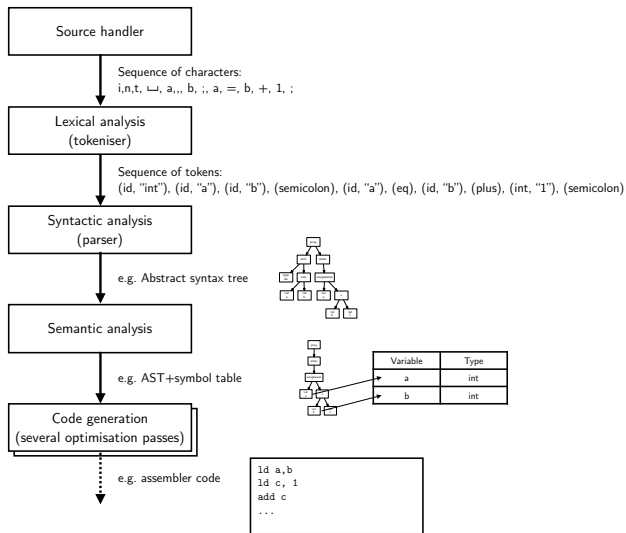
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# High-Level Architecture of a Compiler



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- ▶ Handles input files
- ▶ Provides character-by-character access
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**Result:** Sequence of characters (with positions)



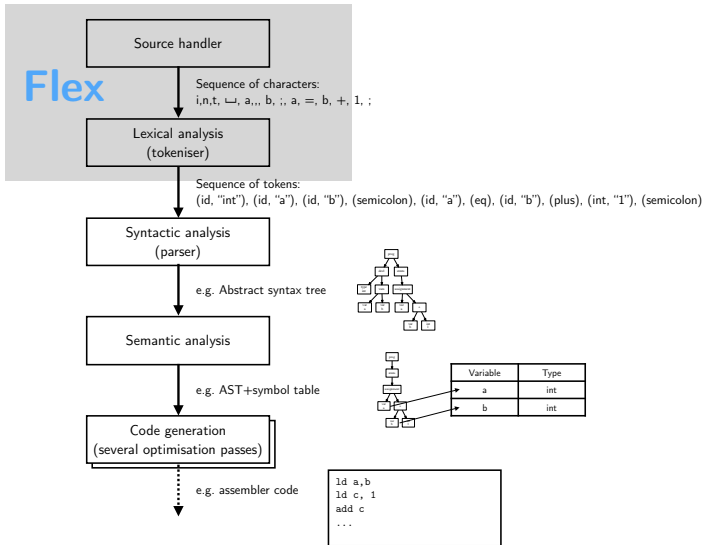
- ▶ Breaks program into **tokens**
- ▶ Typical tokens:
  - ▶ Reserved word (`if`, `while`)
  - ▶ Identifier (`i`, `database`)
  - ▶ Symbols (`{`, `}`, `(`, `)`, `+`, `-`, `...`)

# Lexical Analysis/Scanning

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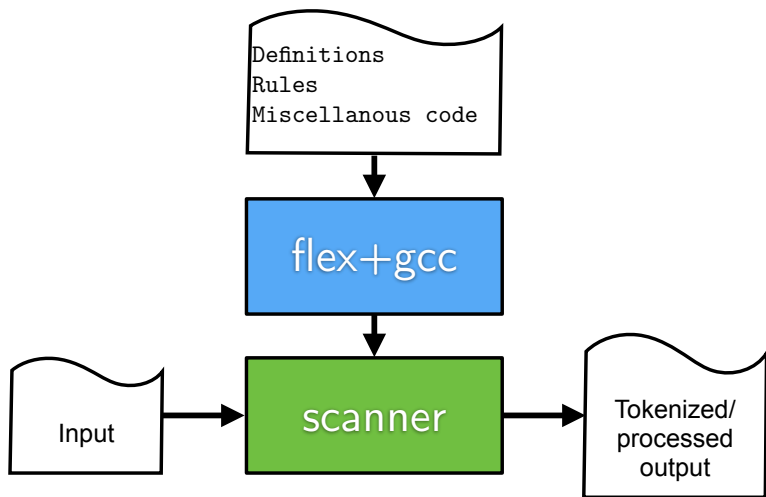
# Automatisation with Flex



# Flex Overview

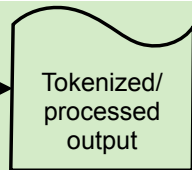
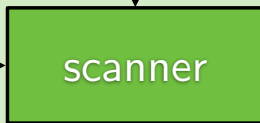
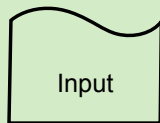
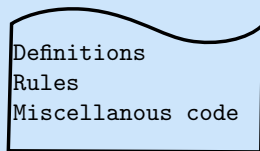
- ▶ Flex is a **scanner generator**
- ▶ Input: Specification of a regular language and what to do with it
  - ▶ Definitions - named regular expressions
  - ▶ Rules - patterns+actions
  - ▶ (miscellaneous support code)
- ▶ Output: Source code of **scanner**
  - ▶ Scans input for patterns
  - ▶ Executes associated actions
  - ▶ Default action: Copy input to output
  - ▶ Interface for higher-level processing: `yylex()` function

# Flex Overview



# Flex Overview

Development time



Execution time

# Flex Example Task

- ▶ Goal: Sum up all numbers in a file, separately for ints and floats
- ▶ Given: A file with numbers and commands
  - ▶ Ints: Non-empty sequences of digits
  - ▶ Floats: Non-empty sequences of digits, followed by decimal dot, followed by (potentially empty) sequence of digits
  - ▶ Command `print`: Print current sums
  - ▶ Command `reset`: Reset sums to 0.
- ▶ At end of file, print sums

# Flex Example Output

## Input

```
12 3.1415
0.33333
print reset
2 11
1.5 2.5 print
1
print 1.0
```

## Output

```
int: 12 ("12")
float: 3.141500 ("3.1415")
float: 0.333330 ("0.33333")
Current: 12 : 3.474830
Reset
int: 2 ("2")
int: 11 ("11")
float: 1.500000 ("1.5")
float: 2.500000 ("2.5")
Current: 13 : 4.000000
int: 1 ("1")
Current: 14 : 4.000000
float: 1.000000 ("1.0")
Final 14 : 5.000000
```



# Basic Structure of Flex Files

- ▶ Flex files have 3 sections
  - ▶ Definitions
  - ▶ Rules
  - ▶ User Code
- ▶ Sections are separated by `%%`
- ▶ Flex files traditionally use the suffix `.l`

## Example Code (definition section)

```
%option noyywrap  
  
DIGIT    [0-9]  
  
%{  
    int intval    = 0;  
    double floatval = 0.0;  
}%  
  
%%
```

## Example Code (rule section)

```
{DIGIT}+    {
    printf( "int:   %d (\"%s\")\n", atoi(yytext), yytext );
    intval += atoi(yytext);
}
{DIGIT}+"."{DIGIT}*    {
    printf( "float: %f (\"%s\")\n", atof(yytext),yytext );
    floatval += atof(yytext);
}
reset {
    intval = 0;
    floatval = 0;
    printf("Reset\n");
}
print {
    printf("Current: %d : %f\n", intval, floatval);
}
\n|. {
    /* Skip */
}
```

## Example Code (user code section)

```
%%  
int main( int argc, char **argv )  
{  
    ++argv, --argc; /* skip over program name */  
    if ( argc > 0 )  
        yyin = fopen( argv[0], "r" );  
    else  
        yyin = stdin;  
  
    yylex();  
  
    printf("Final  %d : %f\n", intval, floatval);  
}
```

## Generating a scanner

```
> flex -t numbers.l > numbers.c
> gcc -c -o numbers.o numbers.c
> gcc numbers.o -o scan_numbers
> ./scan_numbers Numbers.txt
int: 12 ("12")
float: 3.141500 ("3.1415")
float: 0.333330 ("0.33333")
Current: 12 : 3.474830
Reset
int: 2 ("2")
int: 11 ("11")
float: 1.500000 ("1.5")
float: 2.500000 ("2.5")
...
```

## Flexing in detail

```
> flex -tv numbers.l > numbers.c
scanner options: -tvI8 -Cem
37/2000 NFA states
18/1000 DFA states (50 words)
5 rules
Compressed tables always back-up
1/40 start conditions
20 epsilon states, 11 double epsilon states
6/100 character classes needed 31/500 words
of storage, 0 reused
36 state/nextstate pairs created
24/12 unique/duplicate transitions
...
381 total table entries needed
```

# Exercise: Building a Scanner

- ▶ Download the `flex` example and input from <http://www.lehre.dhbw-stuttgart.de/~sschulz/fla2020.html>
- ▶ Build and execute the program:
  - ▶ Generate the scanner with `flex`
  - ▶ Compile/link the C code with `gcc`
  - ▶ Execute the resulting program in the input file

- ▶ Can contain `flex` options
- ▶ Can contain (C) initialization code
  - ▶ Typically `#include()` directives
  - ▶ Global variable definitions
  - ▶ Macros and type definitions
  - ▶ Initialization code is embedded in `%{` and `%}`
- ▶ Can contain definitions of regular expressions
  - ▶ Format: `NAME RE`
  - ▶ Defined `NAMES` can be referenced later



# Regular Expressions in Practice (1)

- ▶ The minimal syntax of REs as discussed before suffices to show their equivalence to finite state machines
- ▶ Practical implementations of REs (e.g. in Flex) use a richer and more powerful syntax
- ▶ Regular expressions in Flex are based on the ASCII alphabet
- ▶ We distinguish between the set of operator symbols

$$O = \{., *, +, ?, -, \sim, |, (, ), [, ], \{, \}, <, >, /, \backslash, \wedge, \$, \}$$

and the set of regular expressions

1.  $c \in \Sigma_{\text{ASCII}} \setminus O \longrightarrow c \in R$
2.  $“.” \in R$   
any character but newline ( $\backslash n$ )

## Regular Expressions in Practice (2)

3.  $x \in \{a, b, f, n, r, t, v\} \longrightarrow \backslash x \in R$   
defines the following control characters
- $\backslash a$  (alert)
  - $\backslash b$  (backspace)
  - $\backslash f$  (form feed)
  - $\backslash n$  (newline)
  - $\backslash r$  (carriage return)
  - $\backslash t$  (tabulator)
  - $\backslash v$  (vertical tabulator)
4.  $a, b, c \in \{0, \dots, 7\} \longrightarrow \backslash abc \in R$  octal representation of a character's ASCII code (e.g.  $\backslash 040$  represents the empty space “ ”)

## Regular Expressions in Practice (3)

5.  $c \in O \longrightarrow \backslash c \in R$   
escaping operator symbols
6.  $r_1, r_2 \in R \longrightarrow r_1 r_2 \in R$   
concatenation
7.  $r_1, r_2 \in R \longrightarrow r_1 | r_2 \in R$   
infix operation using “|” rather than “+”
8.  $r \in R \longrightarrow r^* \in R$   
Kleene star
9.  $r \in R \longrightarrow r^+ \in R$   
(one or more of  $r$ )
10.  $r \in R \longrightarrow r^? \in R$   
optional presence (zero or one  $r$ )

## Regular Expressions in Practice (4)

- $r \in R, n \in \mathbb{N} \rightarrow r\{n\} \in R$   
concatenation of  $n$  times  $r$
- $r \in R; m, n \in \mathbb{N}; m \leq n \rightarrow r\{m, n\} \in R$   
concatenation of between  $m$  and  $n$  times  $r$
- $r \in R \rightarrow \hat{r} \in R$   
 $r$  has to be at the **beginning** of line
- $r \in R \rightarrow r\$ \in R$   
 $r$  has to be at the **end** of line
- $r_1, r_2 \in R \rightarrow r_1/r_2 \in R$   
The same as  $r_1r_2$ , however, only the contents of  $r_1$  is consumed.  
The **trailing context**  $r_2$  can be processed by the next rule.
- $r \in R \rightarrow (r) \in R$   
Grouping regular expressions with brackets.

## 17. Ranges

- $[aeiou] \doteq a|e|i|o|u$
- $[a-z] \doteq a|b|c|\dots|z$
- $[a-zA-Z0-9]$ : alphanumeric characters
- $[\^0-9]$ : all ASCII characters w/o digits

## 18. $[ ] \in R$

empty space

19.  $w \in \{\Sigma_{\text{ASCII}} \setminus \{\backslash, \text{"}\}\}^* \longrightarrow \text{"}w\text{"} \in R$   
verbatim text (no escape sequences)

21.  $r \in R \longrightarrow \sim r \in R$

The `upto` operator matches the **shortest** string ending with  $r$ .

22. predefined character classes

- ▶ `[:alnum:]`    `[:alpha:]`    `[:blank:]`
- ▶ `[:cntrl:]`    `[:digit:]`    `[:graph:]`
- ▶ `[:lower:]`    `[:print:]`    `[:punct:]`
- ▶ `[:space:]`    `[:upper:]`    `[:xdigit:]`

# Regular Expressions in Practice (precedences)

- I. “(”, “)” (strongest)
- II. “\*”, “+”, “?”
- III. concatenation
- IV. “|” (weakest)

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- IV. “|” (weakest)

## Example

$a*b | c+de \doteq ((a*)b) | (((c+)d)e)$



# Regular Expressions in Practice (precedences)

- I. “(”, “)” (strongest)
- II. “\*”, “+”, “?”
- III. concatenation
- IV. “|” (weakest)

## Example

$a*b|c+de \doteq ((a*)b) | (((c+)d)e)$

**Rule of thumb: \*, +, ? bind the smallest possible RE.  
Use () if in doubt!**

# Regular Expressions in Practice (definitions)

- ▶ Assume definition `NAME DEF`
  - ▶ In later REs. `{NAME}` is expanded to `(DEF)`
- ▶ Example:

```
DIGIT  [0-9]
INTEGER {DIGIT}+
PAIR    \({INTEGER}, {INTEGER}\)
```

## Exercise: extended regular expressions

Given the alphabet  $\Sigma_{ascii}$ , how would you express the following practical REs using only the simple REs we have used so far?

1 [a-z]

2 [^0-9]

3 (r)+

4 (r){3}

5 (r){3,7}

6 (r)?

## Example Code (definition section) (revisited)

```
%option noyywrap

DIGIT    [0-9]

%{
    int    intval    = 0;
    double floatval = 0.0;
}%

%%
```

# Rule Section

- ▶ This is the core of the scanner!
- ▶ Rules have the form `PATTERN ACTION`
- ▶ Patterns are regular expressions
  - ▶ Typically use previous definitions
- ▶ There has to be white space between pattern and action
- ▶ Actions are C code
  - ▶ Can be embedded in `{` and `}`
  - ▶ Can be simple C statements
  - ▶ For a token-by-token scanner, must include `return` statement
  - ▶ Inside the action, the variable `yytext` contains the text matched by the pattern
  - ▶ Otherwise: Full input file is processed

## Example Code (rule section) (revisited)

```
{DIGIT}+    {
    printf( "int:   %d (\"%s\")\n", atoi(yytext), yytext );
    intval += atoi(yytext);
}
{DIGIT}+"."{DIGIT}*    {
    printf( "float: %f (\"%s\")\n", atof(yytext),yytext );
    floatval += atof(yytext);
}
reset {
    intval = 0;
    floatval = 0;
    printf("Reset\n");
}
print {
    printf("Current: %d : %f\n", intval, floatval);
}
\n|. {
    /* Skip */
}
```

## User code section

- ▶ Can contain all kinds of code
- ▶ For stand-alone scanner: must include `main()`
- ▶ In `main()`, the function `yylex()` will invoke the scanner
- ▶ `yylex()` will read data from the file pointer `yyin`  
(so `main()` must set it up reasonably)

## Example Code (user code section) (revisited)

```
%%  
int main( int argc, char **argv )  
{  
    ++argv, --argc; /* skip over program name */  
    if ( argc > 0 )  
        yyin = fopen( argv[0], "r" );  
    else  
        yyin = stdin;  
  
    yylex();  
  
    printf("Final   %d : %f\n", intval, floatval);  
}
```



# A comment on comments

- ▶ Comments in Flex are complicated
  - ▶ ...because nearly everything can be a pattern
- ▶ Simple rules:
  - ▶ Use old-style C comments `/* This is a comment */`
  - ▶ Never start them in the first column
  - ▶ Comments are copied into the generated code
  - ▶ Read the manual if you want the dirty details

▶ Flex online:

▶ `https://github.com/westes/flex`

▶ **Manual:** `https://westes.github.io/flex/manual/`

▶ **REs:** `https://westes.github.io/flex/manual/Patterns.html`

▶ Flex online:

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▶ `make` knows flex

- ▶ **Make will automatically generate** `file.o` from `file.l`
- ▶ **Be sure to set** `LEX=flex` to enable flex extensions
- ▶ **Makefile example:**

```
LEX=flex
all: scan_numbers
numbers.o: numbers.l

scan_numbers: numbers.o
    gcc numbers.o -o scan_numbers
```

# Flexercise (1)

A security audit firm needs a tool that scans documents for the following:

- ▶ Email addresses

- ▶ Format: String over `[A-Za-z0-9_~]`, followed by `@`, followed by a domain name according to RFC-1034, <https://tools.ietf.org/html/rfc1034>, Section 3.5 (we only consider the case that the domain name is not empty)

- ▶ (simplified) Web addresses

- ▶ `http://` followed by an RFC-1034 domain name, optionally followed by `:<port>` (where `<port>` is a non-empty sequence of digits), optionally followed by one or several parts of the form `/<str>`, where `<str>` is a non-empty sequence over `[A-Za-z0-9_~]`

## Flexercise (2)

### ▶ Bank account numbers

- ▶ Old-style bank account numbers start with an identifying string, optionally followed by ., optionally followed by :, optionally followed by spaces, followed by a non-empty sequence of up to 10 digits. Identifying strings are `Konto`, `Kto`, `KNr`, `Ktonr`, `Kontonummer`
- ▶ (German) IBANs are strings starting with `DE`, followed by exactly 20 digits. Human-readable IBANs have spaces after every 4 characters (the last group has only 2 characters)

### ▶ Examples:

- ▶ `Rosenda@gidwd-39.at.z8o3rw2.zhv`
- ▶ `http://jzl.j51g.m-x95.vi5/ojlg_i1/72zz_gt68f`
- ▶ `http://iefbottw99.v4gy.zslu9q.zrc2es01nr.dy:8004`
- ▶ `Ktonr. 241524`
- ▶ `DE26959558703965641174`
- ▶ `DE27 0192 8222 4741 4694 55`

## Flexercise (3)

- ▶ Create a programm scanning for the data described above and printing the found items.
- ▶ Example input/output data for Stephan Schulz's lecture can be found under

```
http://wwwlehre.dhbw-stuttgart.de/~sschulz/  
fla2020.html
```

## Flexercise (3)

- ▶ Create a programm scanning for the data described above and printing the found items.
- ▶ Example input/output data for Stephan Schulz's lecture can be found under

`http://www.lehre.dhbw-stuttgart.de/~sschulz/fla2020.html`

End lecture 10

# Outline

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Regular Languages and Finite Automata

Scanners and Flex

**Formal Grammars and Context-Free Languages**

Formal Grammars

The Chomsky Hierarchy

Right-linear Grammars

Context-free Grammars

Push-Down Automata  
Properties of Context-free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

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# Formal Grammars: Motivation

So far, we have seen

- ▶ regular expressions: compact description of regular languages
- ▶ finite automata: recognise words of a regular language

# Formal Grammars: Motivation

So far, we have seen

- ▶ regular expressions: compact description of regular languages
- ▶ finite automata: recognise words of a regular language

Another, more powerful formalism: formal grammars

- ▶ generate words of a language
- ▶ contain a set of rules allowing to replace symbols with different symbols

## Example (Formal grammars)

$$S \rightarrow aA, \quad A \rightarrow bB, \quad B \rightarrow \varepsilon$$

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generates  $ab$  (starting from  $S$ ):  $S \rightarrow aA \rightarrow abB \rightarrow ab$

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generates  $ab$  (starting from  $S$ ):  $S \rightarrow aA \rightarrow abB \rightarrow ab$

$S \rightarrow \varepsilon, \quad S \rightarrow aSb$

generates  $a^n b^n$

## Definition (Grammar according to Chomsky)

A (formal) grammar is a quadruple

$$G = (N, \Sigma, P, S)$$

with

- 1 the set of non-terminal symbols  $N$ ,
- 2 the set of terminal symbols  $\Sigma$ ,
- 3 the set of production rules  $P$  of the form

$$\alpha \rightarrow \beta$$

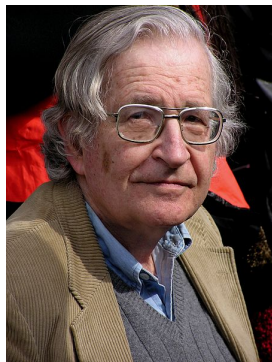
with  $\alpha \in V^*NV^*$ ,  $\beta \in V^*$ ,  $V = N \cup \Sigma$

- 4 the distinguished start symbol  $S \in N$ .



# Noam Chomsky (\*1928)

- ▶ Linguist, philosopher, logician, . . .
- ▶ BA, MA, PhD (1955) at the University of Pennsylvania
- ▶ Mainly teaching at MIT (since 1955)
  - ▶ Also Harvard, Columbia University, Institute of Advanced Studies (Princeton), UC Berkely, . . .
  - ▶ Currently Laureate Professor of Linguistics at University of Arizona, Institute Professor Emeritus at MIT
- ▶ Opposition to Vietnam War, Essay *The Responsibility of Intellectuals*
- ▶ Most cited academic (1980-1992)
- ▶ “World’s top public intellectual” (2005)
- ▶ More than 40 honorary degrees



## Example (C identifiers)

$G = (N, \Sigma, P, S)$  describes C identifiers:

- ▶ alpha-numeric words
- ▶ which must not start with a digit
- ▶ and may contain an underscore (`_`)

# Grammar for C identifiers

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- ▶ which must not start with a digit
- ▶ and may contain an underscore (`_`)

$N = \{S, R, L, D\}$  (start, rest, letter, digit),

$\Sigma = \{a, \dots, z, A, \dots, Z, 0, \dots, 9, \_ \}$ ,

$P = \{$

$S$	$\rightarrow$	$LR _R$
$R$	$\rightarrow$	$LR DR _R \varepsilon$
$L$	$\rightarrow$	$a \dots z A \dots Z$
$D$	$\rightarrow$	$0 \dots 9\}$

# Grammar for C identifiers

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$N = \{S, R, L, D\}$  (start, rest, letter, digit),

$\Sigma = \{a, \dots, z, A, \dots, Z, 0, \dots, 9, \_ \}$ ,

$$P = \left\{ \begin{array}{l} S \rightarrow LR|_R \\ R \rightarrow LR|DR|_R|\varepsilon \\ L \rightarrow a|\dots|z|A|\dots|Z \\ D \rightarrow 0|\dots|9 \end{array} \right.$$

$\alpha \rightarrow \beta_1 | \dots | \beta_n$  is an abbreviation for  $\alpha \rightarrow \beta_1, \dots, \alpha \rightarrow \beta_n$ .

# Formal grammars: derivation, language

## Definition (Derivation, Language of a Grammar)

For a grammar  $G = (N, \Sigma, P, S)$  with  $V = (\Sigma \cup N)$  and words  $x, y \in V^*$ , we say that

$G$  derives  $y$  from  $x$  in one step  $(x \Rightarrow_G y)$  iff

$$\exists u, v, p, q \in V^* : (x = upv) \wedge (p \rightarrow q \in P) \wedge (y = uqv)$$

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Moreover, we say that

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$$\exists w_0, \dots, w_n$$

with  $w_0 = x, w_n = y, w_{i-1} \Rightarrow_G w_i$  for  $i \in \{1, \dots, n\}$

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The language of  $G$  is  $L(G) = \{w \in \Sigma^* \mid S \Rightarrow_G^* w\}$

## Example ( $G_3$ )

Let  $G_3 = (N, \Sigma, P, S)$  with

- ▶  $N = \{S\}$ ,
- ▶  $\Sigma = \{a\}$ ,
- ▶  $P = \{S \rightarrow aS, \quad S \rightarrow \varepsilon\}$ .



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Derivations of  $G_3$  have the general form

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$$S \Rightarrow aS \Rightarrow aaS \Rightarrow \dots \Rightarrow a^n S \Rightarrow a^n$$

The language produced by  $G_3$  is

$$L(G_3) = \{a^n \mid n \in \mathbb{N}\}.$$

## Example ( $G_2$ )

Let  $G_2 = (N, \Sigma, P, S)$  with

- ▶  $N = \{S\}$ ,
- ▶  $\Sigma = \{a, b\}$ ,
- ▶  $P = \{S \rightarrow aSb, S \rightarrow \varepsilon\}$

# Grammars and derivations (cont')

## Example ( $G_2$ )

Let  $G_2 = (N, \Sigma, P, S)$  with

- ▶  $N = \{S\}$ ,
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Derivations of  $G_2$ :

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Derivations of  $G_2$ :

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow \dots \Rightarrow a^n S b^n \Rightarrow a^n b^n.$$

$$L(G_2) = \{a^n b^n \mid n \in \mathbb{N}\}.$$

**Reminder:  $L(G_2)$  is not regular!**

## Example ( $G_1$ )

Let  $G_1 = (N, \Sigma, P, S)$  with

▶  $N = \{S, B, C\}$ ,

▶  $\Sigma = \{a, b, c\}$ ,

▶  $P$ :

$$S \rightarrow aSBC \quad 1$$

$$S \rightarrow aBC \quad 2$$

$$CB \rightarrow BC \quad 3$$

$$aB \rightarrow ab \quad 4$$

$$bB \rightarrow bb \quad 5$$

$$bC \rightarrow bc \quad 6$$

$$cC \rightarrow cc \quad 7$$

# Exercise: Derivations in $G_1$

## Example ( $G_1$ )

Let  $G_1 = (N, \Sigma, P, S)$  with

- ▶  $N = \{S, B, C\}$ ,
- ▶  $\Sigma = \{a, b, c\}$ ,
- ▶  $P$ :

$$S \rightarrow aSBC \quad 1$$

$$S \rightarrow aBC \quad 2$$

$$CB \rightarrow BC \quad 3$$

$$aB \rightarrow ab \quad 4$$

$$bB \rightarrow bb \quad 5$$

$$bC \rightarrow bc \quad 6$$

$$cC \rightarrow cc \quad 7$$

- ▶ Give derivations for 3 different words in  $L(G_1)$
- ▶ Can you characterize  $L(G_1)$ ?



# Grammars and derivations (cont.)

Derivations of  $G_1$ :

$$\begin{aligned} S &\Rightarrow_1 aSBC \Rightarrow_1 aaSBCBC \Rightarrow_1 \cdots \Rightarrow_1 a^{n-1}S(BC)^{n-1} \Rightarrow_2 a^n(BC)^n \\ &\Rightarrow_3^* a^nB^nC^n \Rightarrow_{4,5}^* a^nb^nC^n \Rightarrow_{6,7}^* a^nb^nc^n \end{aligned}$$

# Grammars and derivations (cont.)

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$$L(G_1) = \{a^nb^nc^n \mid n \in \mathbb{N}; n > 0\}.$$

# Grammars and derivations (cont.)

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$$L(G_1) = \{a^nb^nc^n \mid n \in \mathbb{N}; n > 0\}.$$

- ▶ These three derivation examples represent different classes of grammars or languages characterized by different properties.
- ▶ A widely used classification scheme of formal grammars and languages is the [Chomsky hierarchy](#) (1956).

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# The Chomsky hierarchy (0)

## Definition (Grammar of type 0)

Every Chomsky grammar  $G = (N, \Sigma, P, S)$  is of **Type 0** or **unrestricted**.

# The Chomsky hierarchy (1)

## Definition (context-sensitive grammar)

A Chomsky grammar  $G = (N, \Sigma, P, S)$  is of is **Type 1** (**context-sensitive**) if all productions are of the form

$$\alpha \rightarrow \beta \quad \text{with} \quad |\alpha| \leq |\beta|$$

Exception: the rule  $S \rightarrow \varepsilon$  is allowed if  $S$  does not appear on the right-hand side of any rule

# The Chomsky hierarchy (1)

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Exception: the rule  $S \rightarrow \varepsilon$  is allowed if  $S$  does not appear on the right-hand side of any rule

- ▶ Rules never derive shorter words
  - ▶ except possibly for the empty word in the first step

## Context-sensitive?

- ▶ What is **context-sensitive** about grammars as defined on the previous slides?



# Context-sensitive?

- ▶ What is **context-sensitive** about grammars as defined on the previous slides?
- ▶ Grammars of the type defined on the last slide were originally called **monotonic** or **non-contracting** by Chomsky

# Context-sensitive?

- ▶ What is **context-sensitive** about grammars as defined on the previous slides?
- ▶ Grammars of the type defined on the last slide were originally called **monotonic** or **non-contracting** by Chomsky
- ▶ **Context-sensitive** grammars additionally had to satisfy the property that all rules are of the form

$$\alpha_1 A \alpha_2 \rightarrow \alpha_1 \beta \alpha_2 \text{ with } A \in N; \alpha_1, \alpha_2 \in V^*, \beta \in VV^*$$

- ▶ rule application can depend on a context  $\alpha_1, \alpha_2$
- ▶ context cannot be modified (or moved)
- ▶ only **one NTS** can be modified

# Context-sensitive vs. monotonic grammars

- ▶ Every monotonic grammar can be rewritten as context-sensitive
  - ▶  $AB \rightarrow BA$  is not context-sensitive, but can be replaced by  $AB \rightarrow AY, AY \rightarrow XY, XY \rightarrow XA, XA \rightarrow BA$ , which is
  - ▶ if terminal symbols are involved: replace  $S \rightarrow aB \rightarrow ba$  with  $S \rightarrow N_a B \rightarrow \dots N_b N_a \rightarrow b N_a \rightarrow ba$
- ▶ Since context preservation is irrelevant for the language class, we drop the context requirement for this lecture
- ▶ Since the term “context-sensitive” is generally used in the literature, we stick with this term (for both grammars and languages), even if we only require non-monotonicity

## The Chomsky hierarchy (2)

### Definition (context-free grammar)

A Chomsky grammar  $G = (N, \Sigma, P, S)$  is of is **Type 2 (context-free)** if all productions are of the form

$$A \rightarrow \beta \text{ with } A \in N; \beta \in V^*$$

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### Definition (context-free grammar)

A Chomsky grammar  $G = (N, \Sigma, P, S)$  is of is **Type 2 (context-free)** if all productions are of the form

$$A \rightarrow \beta \text{ with } A \in N; \beta \in V^*$$

- ▶ Only single non-terminals are replaced
  - ▶ independent of their context
- ▶ Contracting rules are **allowed!**
  - ▶ context-free grammars are **not** a subset of context-sensitive grammars
  - ▶ but: context-free **languages** are a subset of context-sensitive **languages**
  - ▶ reason: contracting rules can be removed from context-free grammars, but not from context-sensitive ones

## The Chomsky hierarchy (3)

### Definition (right-linear grammar)

A Chomsky grammar  $G = (N, \Sigma, P, S)$  is of **Type 3** (right-linear or regular) if all productions are of the form

$$A \rightarrow aB$$

with  $A \in N$ ;  $B \in N \cup \{\epsilon\}$ ;  $a \in \Sigma \cup \{\epsilon\}$

## The Chomsky hierarchy (3)

### Definition (right-linear grammar)

A Chomsky grammar  $G = (N, \Sigma, P, S)$  is of **Type 3** (right-linear or regular) if all productions are of the form

$$A \rightarrow aB$$

with  $A \in N$ ;  $B \in N \cup \{\epsilon\}$ ;  $a \in \Sigma \cup \{\epsilon\}$

- ▶ only one NTS on the left
- ▶ on the right: one TS, one NTS, both, or neither
- ▶ analogy with automata is obvious

## Definition (language classes)

A language is called

recursively enumerable, context-sensitive, context-free, or regular,

if it can be generated by a

unrestricted, context-sensitive, context-free, or regular

grammar, respectively.

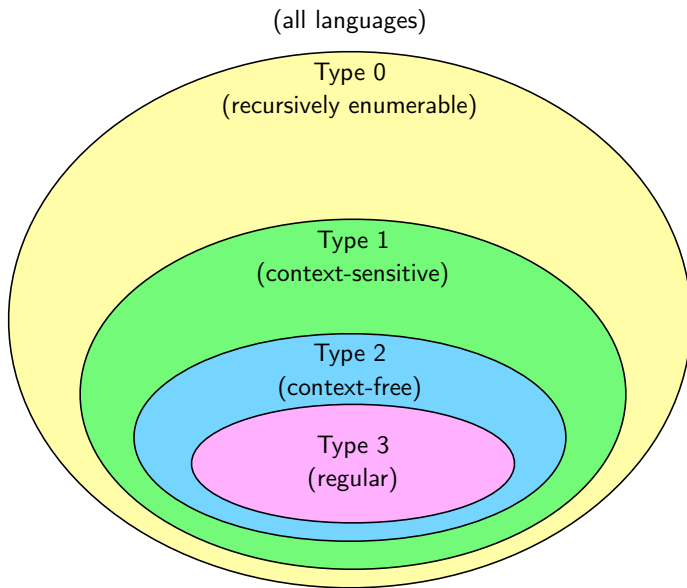


# Formal grammars vs. formal languages vs. machines

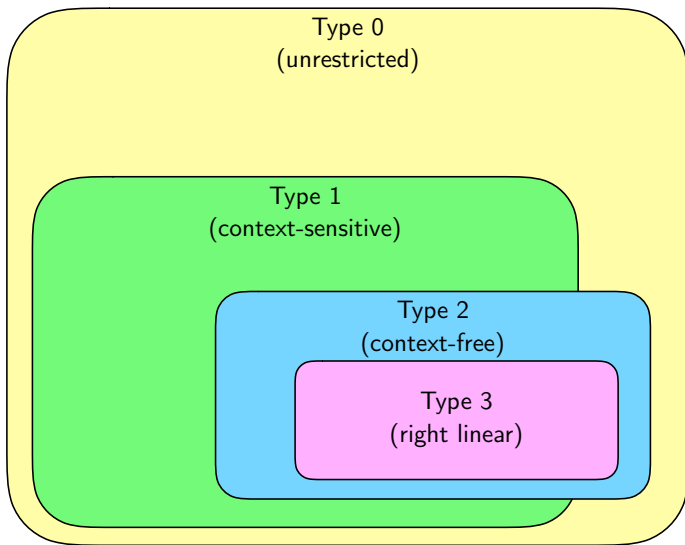
For each grammar/language type, there is a corresponding type of machine model:

grammar	language	machine
Type 0 unrestricted	recursively enumerable	Turing machine
Type 1	context-sensitive	linear-bounded non-deterministic Turing machine
Type 2	context-free	non-deterministic pushdown automaton
Type 3 right linear	regular	finite automaton

# The Chomsky Hierarchy for Languages



# The Chomsky Hierarchy for Grammars



# The Chomsky hierarchy: examples

## Example (C identifiers revisited)

$$S \rightarrow LR\_R$$
$$R \rightarrow LR|DR\_R|\varepsilon$$
$$L \rightarrow a|\dots|z|A|\dots|Z$$
$$D \rightarrow 0|\dots|9$$

# The Chomsky hierarchy: examples

## Example (C identifiers revisited)

$$S \rightarrow LR\_R$$

$$R \rightarrow LR|DR\_R|\varepsilon$$

$$L \rightarrow a|\dots|z|A|\dots|Z$$

$$D \rightarrow 0|\dots|9$$

This grammar is context-free but not regular.

# The Chomsky hierarchy: examples

## Example (C identifiers revisited)

$$S \rightarrow LR\_R$$

$$R \rightarrow LR|DR\_R|\varepsilon$$

$$L \rightarrow a|\dots|z|A|\dots|Z$$

$$D \rightarrow 0|\dots|9$$

This grammar is context-free but not regular.

An equivalent regular grammar:

$$S \rightarrow AR|\dots|ZR|aR|\dots|zR|\_R$$

$$R \rightarrow AR|\dots|ZR|aR|\dots|zR|0R|\dots|9R|\_R|\varepsilon$$

# The Chomsky hierarchy: examples revisited

Returning to the three derivation examples:

- ▶  $G_3$  with  $P = \{S \rightarrow aS, S \rightarrow \varepsilon\}$ 
  - ▶  $G_3$  is regular.
  - ▶ So is the produced language  $L_3 = \{a^n \mid n \in \mathbb{N}\}$ .
- ▶  $G_2$  with  $P = \{S \rightarrow aSb, S \rightarrow \varepsilon\}$ 
  - ▶  $G_2$  is context-free.
  - ▶ So is the produced language  $L_2 = \{a^n b^n \mid n \in \mathbb{N}\}$ .
- ▶  $G_1$  with  $P = \{S \rightarrow aSBC, S \rightarrow aBC, CB \rightarrow BC, \dots\}$ 
  - ▶  $G_1$  is context-sensitive.
  - ▶ So is the produced language  $L_1 = \{a^n b^n c^n \mid n \in \mathbb{N}; n > 0\}$ .

# The Chomsky hierarchy: exercises

**1** Let  $G = (N, \Sigma, P, S)$  with

- ▶  $N = \{S, A, B\}$ ,
- ▶  $\Sigma = \{a\}$ ,
- ▶  $P :$

$$\begin{array}{ll} S \rightarrow \epsilon & 1 \\ S \rightarrow ABA & 2 \\ AB \rightarrow aa & 3 \\ aA \rightarrow aaaA & 4 \\ A \rightarrow a & 5 \end{array}$$

- a) What is  $G$ 's highest type?
- b) Show how  $G$  derives the word  $aaaaa$ .
- c) Formally describe the language  $L(G)$ .
- d) Define a regular grammar  $G'$  equivalent to  $G$ .



## The Chomsky hierarchy: exercises (cont.)

- 2 An **octal constant** is a finite sequence of digits starting with 0 followed by at least one digit ranging from 0 to 7. Define a regular grammar encoding exactly the set of possible octal constants.

# The Chomsky hierarchy: exercises (cont.)

**3** Let  $G = (N, \Sigma, P, S)$  with

▶  $N = \{S, A, B\},$

▶  $\Sigma = \{a, b, t\},$

▶  $P: S \rightarrow aAS \quad 1$

$S \rightarrow bBS \quad 2$

$S \rightarrow t \quad 3$

$At \rightarrow ta \quad 4$

$Bt \rightarrow tb \quad 5$

$Aa \rightarrow aA \quad 6$

$Ab \rightarrow bA \quad 7$

$Ba \rightarrow aB \quad 8$

$Bb \rightarrow bB \quad 9$

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b) Formally describe the language  $L(G)$ .

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End lecture 11

# Outline

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Regular Languages and Finite Automata

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**Formal Grammars and Context-Free Languages**

Formal Grammars

The Chomsky Hierarchy

**Right-linear Grammars**

Context-free Grammars

Push-Down Automata  
Properties of Context-free Languages

Parsers and Bison

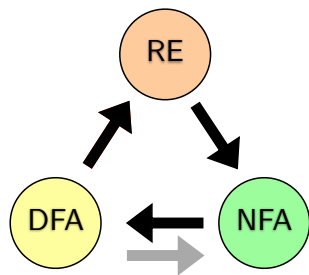
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Bonus Exercises

Selected Solutions

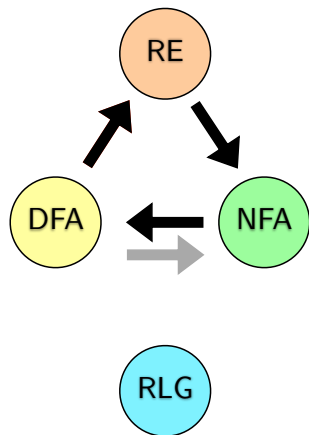
# Right-linear grammars and regular languages



► We know:

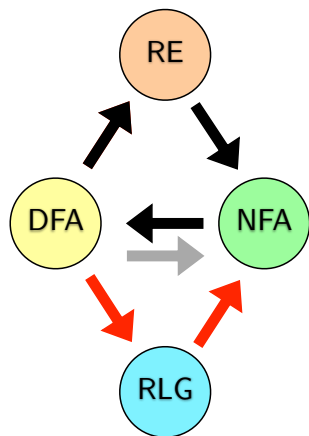
- Regular expression can be converted to NFAs
- NFAs can be converted to DFAs
- DFAs can be described by regular expressions
- Hence NFA, DFA, RE are equivalent and all describe the class of **regular languages**

# Right-linear grammars and regular languages



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# Right-linear grammars and regular languages



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  - ▶ DFAs can be described by regular expressions
  - ▶ Hence NFA, DFA, RE are equivalent and all describe the class of **regular languages**
- ▶ Now: Right-linear grammars
- ▶ We show:
  - ▶ DFAs can be converted into right-linear grammars
  - ▶ Right-linear grammars can be converted into NFAs

# Regular languages and right-linear grammars

## Theorem (right-linear grammars and regular languages)

*The class of regular languages (generated by regular expressions, accepted by finite automata) is exactly the class of languages generated by right-linear grammars.*



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## Proof.

We constructively prove the theorem by providing algorithms to...

- ▶ Convert a DFA to a right-linear grammar
- ▶ Convert right-linear grammar to NFA

... such that the languages of grammar and automaton are the same. □

# DFA $\rightsquigarrow$ right-linear grammar

Algorithm for transforming a DFA

$$\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$$

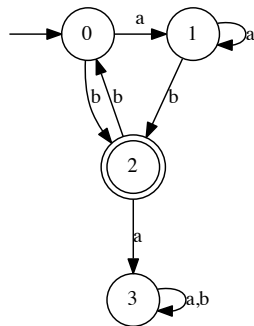
into a grammar

$$G = (N, \Sigma, P, S)$$

- ▶  $N = Q$
- ▶  $S = q_0$
- ▶  $P = \{p \rightarrow aq \mid ((p, a), q) \in \delta\} \cup \{p \rightarrow \varepsilon \mid p \in F\}$

## Exercise: DFA to right-linear grammar

Consider the following DFA  $\mathcal{A}$ :



- Give a formal definition of  $\mathcal{A}$
- Generate a right-linear grammar  $G$  with  $L(G) = L(\mathcal{A})$

# Right-linear grammar $\rightsquigarrow$ NFA

Algorithm for transforming a grammar

$$G = (N, \Sigma, P, S)$$

into an NFA

$$\mathcal{A} = (Q, \Sigma, \Delta, q_0, F)$$

- ▶  $Q = N \cup \{q_f\}$  ( $q_f \notin N$ )
- ▶  $q_0 = S$
- ▶  $F = \{q_f\}$
- ▶  $\Delta = \{(A, c, B) \mid A \rightarrow cB \in P\} \cup$   
 $\{(A, c, q_f) \mid A \rightarrow c \in P\} \cup$   
 $\{(A, \varepsilon, B) \mid A \rightarrow B \in P\} \cup$   
 $\{(A, \varepsilon, q_f) \mid A \rightarrow \varepsilon \in P\}$

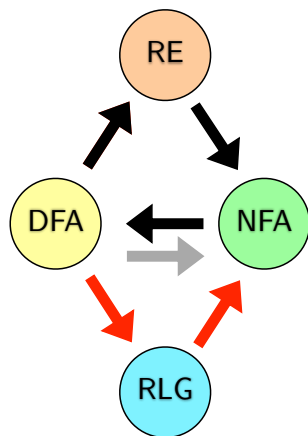
## Exercise: right-linear grammar to NFA

Transform the grammar  $G = (\{S, A, B\}, \{a, b\}, P, S)$  into an NFA.

$$\begin{aligned}P : \quad S &\rightarrow aB \mid \varepsilon \\A &\rightarrow aB \mid b \\B &\rightarrow A\end{aligned}$$

Which language is generated by  $G$ ?

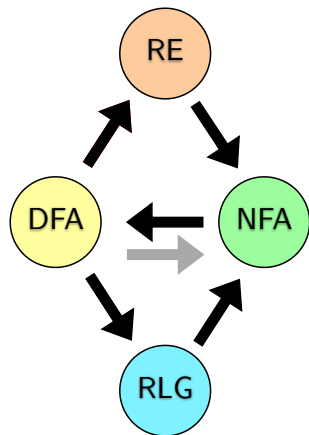
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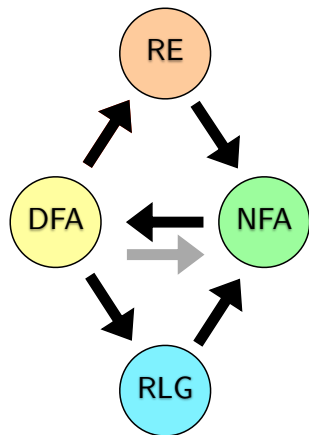
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- DFAs can be converted into right-linear grammars
- Right-linear grammars can be converted into NFAs

# Right-linear grammars and regular languages



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- Regular expression can be converted to NFAs
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- Hence NFA, DFA, RE are equivalent and all describe the class of **regular languages**
- DFAs can be converted into right-linear grammars
- Right-linear grammars can be converted into NFAs

REs, DFAs, NFAs, Right-linear grammars are all equivalent formalisms to describe the class of regular languages!



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# Context-free grammars

- ▶ Reminder:  $G = (N, \Sigma, P, S)$  is context-free if all rules in  $P$  are of the form  $A \rightarrow \beta$  with
  - ▶  $A \in N$  and
  - ▶  $\beta \in (\Sigma \cup N)^*$
- ▶ Context-free languages/grammars are highly relevant for practical applications
  - ▶ Core of most programming languages
  - ▶ XML
  - ▶ Algebraic expressions
  - ▶ Many aspects of human language

# Grammars: equivalence and normal forms

## Definition (equivalence)

Two grammars are called **equivalent** if they generate the same language.

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We will now compute grammars that are equivalent to some given context-free grammar  $G$  but have “nicer” properties

- ▶ **Reduced** grammars contain no unproductive symbols
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- ▶ **Reduced** grammars contain no unproductive symbols
- ▶ Grammars in **Chomsky normal form** support efficient decision of the **word problem**

I.e. grammars in CNF allow efficient parsing of arbitrary context-free languages!

## Definition (reduced)

Let  $G = (N, \Sigma, P, S)$  be a context-free grammar.

- ▶  $A \in N$  is called **terminating** if  $A \Rightarrow_G^* w$  for some  $w \in \Sigma^*$ .
- ▶  $A \in N$  is called **reachable** if  $S \Rightarrow_G^* uAv$  for some  $u, v \in V^*$ .
- ▶  $G$  is called **reduced** if  $N$  contains only reachable and terminating symbols.

# Terminating and reachable symbols

The terminating symbols can be computed as follows:

- 1  $T := \{A \in N \mid \exists w \in \Sigma^* : A \rightarrow w \in P\}$
- 2 add all symbols  $M$  to  $T$  with a rule  $M \rightarrow D$  with  $D \in (\Sigma \cup T)^*$
- 3 repeat step 2 until no further symbols can be added

Now  $T$  contains exactly the terminating symbols.

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Now  $T$  contains exactly the terminating symbols.

The reachable symbols can be computed as follows:

- 1  $R := \{S\}$
- 2 for every  $A \in R$ , add all symbols  $M$  with a rule  $A \rightarrow V^*MV^*$
- 3 repeat step 2 until no further symbols can be added

Now  $R$  contains exactly the reachable symbols.



# Reducing context-free grammars

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- 1 generate the grammar  $G_T$  by removing all **non-terminating** symbols (and rules containing them) from  $G$
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Sequence is important: symbols become unreachable if they only appear together with non-terminating symbols.

## Example

Let  $G = (N, \Sigma, P, S)$  with

▶  $N = \{S, A, B, C, T\}$ ,

▶  $\Sigma = \{a, b, c\}$ ,

▶  $P$  :

$$S \rightarrow T|B|C$$

$$T \rightarrow AB$$

$$A \rightarrow a$$

$$B \rightarrow bB$$

$$C \rightarrow c$$

# Reachable and terminating symbols

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▶  $P :$   $S \rightarrow T|B|C$

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▶ terminating symbols in  $G$ :  $C, A, S \rightsquigarrow G_T$

▶ reachable symbols in  $G_T$ :  $S, C \rightsquigarrow G_r$

▶ note:  $A$  is still reachable in  $G$ !

## Exercise: reducing grammars

Compute the reduced grammar  $G = (N, \Sigma, P, S)$  for the following grammar  $G' = (N', \Sigma, P', S)$ :

1  $N' = \{S, A, B, C, D\},$

2  $\Sigma = \{a, b\},$

3  $P' :$

$$S \rightarrow A|aS|B$$

$$A \rightarrow a$$

$$A \rightarrow AS$$

$$A \rightarrow Ba$$

$$B \rightarrow Ba$$

$$C \rightarrow Da$$

$$D \rightarrow Cb$$

$$D \rightarrow a$$

# Chomsky normal form

Reduced grammars can be further modified to allow for an efficient decision procedure for the word problem.

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## Definition (CNF)

A context-free grammar  $(N, \Sigma, P, S)$  is in **Chomsky normal form** if all rules are of the kind

- ▶  $N \rightarrow a$  with  $a \in \Sigma$
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Transformation of a reduced grammar into CNF:

- 1 remove  $\varepsilon$ -productions
- 2 remove chain rules ( $A \rightarrow B$ )
- 3 introduce auxiliary symbols

# Removal of $\varepsilon$ -productions

## Theorem ( $\varepsilon$ -free grammar)

*Every context-free grammar can be transformed into an equivalent cf. grammar that does not contain rules of the kind  $A \rightarrow \varepsilon$  (except  $S \rightarrow \varepsilon$  if  $S$  does not appear on the rhs).*

# Removal of $\varepsilon$ -productions

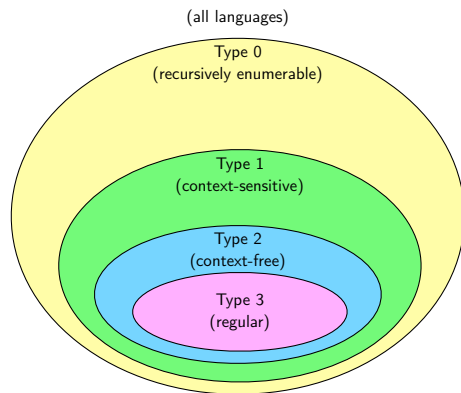
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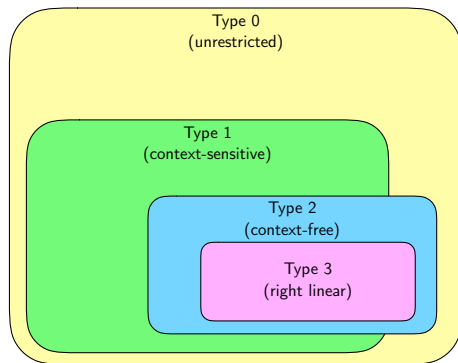
- 1 let  $E = \{A \in N \mid A \rightarrow \varepsilon \in P\}$
- 2 add all symbols  $B$  to  $E$  for which there is a rule  $B \rightarrow \beta$  with  $\beta \in E^*$
- 3 repeat step 2 until no further symbols can be added
- 4 for every rule  $C \rightarrow \beta_1 B \beta_2$  with  $B \in E$ 
  - ▶ add rule  $C \rightarrow \beta_1 \beta_2$  to  $P$
  - ▶ repeat this process until no new rules are added
- 5 remove all rules  $A \rightarrow \varepsilon$  from  $P$
- 6 if  $S \in E$ 
  - ▶ use a new start symbol  $S_0$
  - ▶ add rules  $S_0 \rightarrow \varepsilon | S$

# Interlude: Chomsky-Hierarchy for Grammars (again)



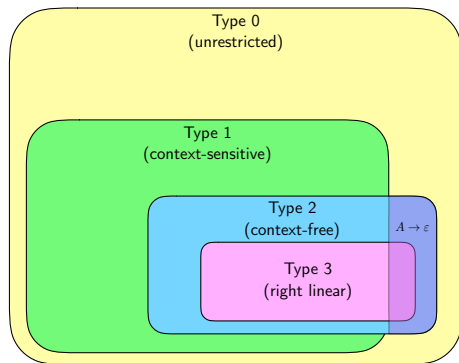
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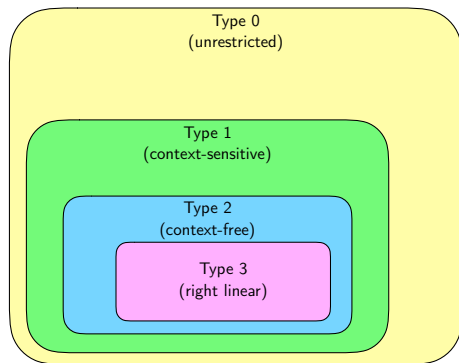
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- ▶ Not quite true for grammars:
  - ▶  $A \rightarrow \varepsilon$  allowed in context-free/regular grammars, not in context-sensitive grammars
- ▶ Eliminating  $\varepsilon$ -productions removes this discrepancy!

End lecture 12

# Removal of chain rules

## Theorem (chain rules)

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(this can be done iteratively, as shown previously)
- 2 remove  $A \rightarrow C$  for any  $C \in N$  from  $P$
- 3 add the following production rules to  $P$   
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## Example

$A \rightarrow a|B; \quad B \rightarrow bb|C; \quad C \rightarrow ccc$

is equivalent to

$A \rightarrow a|bb|ccc; \quad B \rightarrow bb|ccc; \quad C \rightarrow ccc$

# Chomsky normal form

Reminder: Chomsky normal form

A context-free grammar  $(N, \Sigma, P, S)$  is in CNF if all rules are of the kind

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## Theorem (transformation into Chomsky normal form)

*Every context free grammar can be transformed into an equivalent cf. grammar in Chomsky normal form.*

# Algorithm for computing Chomsky normal form

- 1 remove  $\varepsilon$  rules
- 2 remove chain rules
- 3 compute reduced grammar
  - 1 remove non-terminating symbols
  - 2 remove unreachable symbols
- 4 for all rules  $A \rightarrow w$  with  $w \notin \Sigma$ :
  - ▶ for all  $a \in \Sigma$  replace all occurrences of  $a$  in  $w$  by a new non-terminal symbol  $X_a$
  - ▶ add rules  $X_a \rightarrow a$
- 5 replace rules of the form  $A \rightarrow B_1 B_2 \dots B_n$  with  $n > 2$  with rules

$$\begin{aligned}A &\rightarrow B_1 C_1 \\C_1 &\rightarrow B_2 C_2 \\&\vdots \\C_{n-2} &\rightarrow B_{n-1} B_n\end{aligned}$$

where the  $C_i$  are new non-terminals

## Exercise: transformation into CNF

Compute the Chomsky normal form of the following grammar:

$$G = (N, \Sigma, P, S)$$

- ▶  $N = \{S, A, B, C, D, E\}$
- ▶  $\Sigma = \{a, b\}$
- ▶  $P$ :

$$S \rightarrow AB|SB|BDE$$

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Solution

# Chomsky NF: purpose

Why transform  $G$  into Chomsky NF?

- ▶ in a context-free grammar, derivations can have arbitrary length
  - ▶ if there are contracting rules, a derivation of  $w$  can contain words longer than  $w$
  - ▶ if there are chain rules ( $C \rightarrow B; B \rightarrow C$ ), a derivation of  $w$  can contain arbitrarily many steps
- ▶ **word problem** is difficult to decide
- ▶ if  $G$  is in CNF, for a word of length  $n$ , a derivation has  $2n - 1$  steps:
  - ▶  $n - 1$  rule applications  $A \rightarrow BC$
  - ▶  $n$  rule applications  $A \rightarrow a$
- ▶ word problem can be decided by checking all derivations of length  $2n - 1$



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- ▶ if  $G$  is in CNF, for a word of length  $n$ , a derivation has  $2n - 1$  steps:
  - ▶  $n - 1$  rule applications  $A \rightarrow BC$
  - ▶  $n$  rule applications  $A \rightarrow a$
- ▶ word problem can be decided by checking all derivations of length  $2n - 1$
- ▶ That's still plenty of derivations!

# Chomsky NF: purpose

Why transform  $G$  into Chomsky NF?

- ▶ in a context-free grammar, derivations can have arbitrary length
  - ▶ if there are contracting rules, a derivation of  $w$  can contain words longer than  $w$
  - ▶ if there are chain rules ( $C \rightarrow B; B \rightarrow C$ ), a derivation of  $w$  can contain arbitrarily many steps
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**More efficient algorithm: Cocke-Younger-Kasami (CYK)**

# Cocke-Younger-Kasami algorithm

- ▶ Efficient algorithm to decide the word problem for context-free grammars
- ▶ Core ideas independently developed by
  - ▶ John Cocke (1925—2002): *Programming languages and their compilers: Preliminary notes*, 1970 (with Jacob T. Schwartz)
  - ▶ Daniel H. Younger (??–): *Recognition and parsing of context-free languages in time  $n^3$* , 1967
  - ▶ Tadao Kasami (1930–2007) *An efficient recognition and syntax-analysis algorithm for context-free languages*, 1965
- ▶ Complexity:  $O(|w|^3 \cdot |G|)$
- ▶ Can provide *all* ways to parse/generate a word
- ▶ Extends to probabilistic parsing

## CYK algorithm: idea

Decide the word problem for a context-free grammar  $G$  in Chomsky NF and a word  $w$ .

- ▶ find out which NTS are needed in the end to produce the TS for  $w$  (using production rules  $A \rightarrow a$ ).
- ▶ iteratively find all NTS that can generate the required sequence of NTS (using production rules  $A \rightarrow BC$ ).
- ▶ if  $S$  can produce the required sequence,  $w \in L(G)$  holds.

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- ▶ operates on a table.
- ▶ field in row  $i$  and column  $j$  contains all NTS that can generate the target word from character  $i$  through  $j$ .

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**Example of dynamic programming!**

# CYK algorithm: example

$S \rightarrow a$

$B \rightarrow b$

$B \rightarrow c$

$S \rightarrow SA$

$A \rightarrow BS$

$B \rightarrow BB$

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$i \setminus j$	1	2	3	4	5	6
1						
2						
3						
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# CYK: method

- ▶ Core idea: Bottom-Up **dynamic programming**
  - ▶ For all subwords of  $w$  compute the set of **all** non-terminal symbols from which it can be derived
  - ▶ Results are stored in the table: Field  $i, j$  stores the non-terminals from which  $w[i] \dots w[j]$  can be derived
  - ▶ Start with words of length 1 (on the main diagonal)
    - ▶ Only rules of the form  $N \rightarrow t$  relevant
  - ▶ Fill successive diagonals (going right/up)
    - ▶ Remember: Each field represents the NTS which generate a subword!
    - ▶ Consider all splits of the subword (by going left to right in the row, top to bottom in the column)
    - ▶ Consider all combinations of non-terminals for each split
    - ▶ Only rules of the form  $N \rightarrow XY$  relevant: If  $XY$  is a combination generating  $w[i] \dots w[j]$ , add  $N$  at  $(i, j)$ .
  - ▶ If  $S$  generates  $w[1] \dots w[|w|](= w)$ , then  $w \in L(G)$

# CYK: formal algorithm

**for**  $i := 1$  to  $n$  **do**

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$j := i + d$

$N_{ij} := \{\}$



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**for**  $i := 1$  to  $n - d$  **do**

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**for**  $k := i$  to  $j - 1$  **do**

$N_{ij} := N_{ij} \cup \{A \mid A \rightarrow BC \in P; B \in N_{ik}; C \in N_{(k+1)j}\}$

# CYK algorithm: exercise

Consider the grammar  
 $G = (N, \Sigma, P, S)$  from the previous  
exercise

- ▶  $N = \{S, A, B, C\}$
- ▶  $\Sigma = \{a, b\}$

$$\begin{aligned}P : \quad S &\rightarrow AB|SB|BDE \\ A &\rightarrow Aa \\ B &\rightarrow bB|BaB|ab \\ C &\rightarrow SB \\ D &\rightarrow E \\ E &\rightarrow \varepsilon\end{aligned}$$

Use the CYK algorithm to determine if the following words can be generated by  $G$ :

- a)  $w_1 = babaab$
- b)  $w_2 = abba$

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$$\begin{aligned}P : \quad S &\rightarrow SB|BC_1|X_bB|X_aX_b \\ B &\rightarrow BC_1|X_bB|X_aX_b \\ C_1 &\rightarrow X_aB \\ X_a &\rightarrow a \\ X_b &\rightarrow b\end{aligned}$$

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Solution

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# Outline

Introduction

Regular Languages and Finite Automata

Scanners and Flex

**Formal Grammars and Context-Free Languages**

Formal Grammars

The Chomsky Hierarchy

Right-linear Grammars

Context-free Grammars

**Push-Down Automata**

Properties of Context-free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

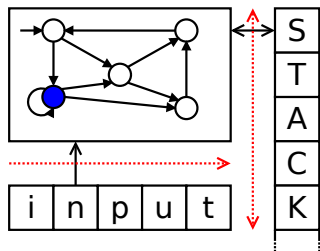
Selected Solutions

# Pushdown automata: motivation

- ▶ DFAs/NFAs are weaker than context-free grammars
- ▶ to accept languages like  $a^n b^n$ , an **unlimited storage component** is needed
- ▶ **Pushdown automata** have an unlimited **stack**
  - ▶ LIFO: last in, first out
  - ▶ only top symbol can be read
  - ▶ arbitrary amount of symbols can be added to the top



# PDA: conceptual model



- ▶ extends FA by **unlimited stack**:
  - ▶ transitions can read and **write** stack
  - ▶ only at the top
  - ▶ **stack alphabet**  $\Gamma$
  - ▶ **LIFO**: last in, first out
- ▶ acceptance condition
  - ▶ **empty stack** after reading input
  - ▶ no final states needed
- ▶ commonalities with FA:
  - ▶ read input from left to right
  - ▶ set of states, input alphabet
  - ▶ initial state

# PDA transitions

$$\Delta \subseteq Q \times \Sigma \cup \{\epsilon\} \times \Gamma \times \Gamma^* \times Q$$

- ▶ PDA is in a state
- ▶ can read next input character or nothing
- ▶ must read (and remove) top stack symbol
- ▶ can write arbitrary amount of symbols on top of stack
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A transition  $(p, c, A, BC, q)$  can be written as follows:

$$p \quad c \quad A \quad \rightarrow \quad BC \quad q$$

# Pushdown automata: definition

## Definition (pushdown automaton)

A **pushdown automaton** (PDA) is a 6-tuple  $(Q, \Sigma, \Gamma, \Delta, q_0, Z_0)$  where

- ▶  $Q, \Sigma, q_0$  are defined as for NFAs.
- ▶  $\Gamma$  is the stack alphabet
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A **configuration** of a PDA is a triple  $(q, w, \gamma)$  where

- ▶  $q$  is the current state
- ▶  $w$  is the input yet unread
- ▶  $\gamma$  is the current stack content

# Pushdown automata: definition

## Definition (pushdown automaton)

A **pushdown automaton** (PDA) is a 6-tuple  $(Q, \Sigma, \Gamma, \Delta, q_0, Z_0)$  where

- ▶  $Q, \Sigma, q_0$  are defined as for NFAs.
- ▶  $\Gamma$  is the stack alphabet
- ▶  $Z_0$  is the initial stack symbol
- ▶  $\Delta \subseteq Q \times \Sigma \cup \{\varepsilon\} \times \Gamma \times \Gamma^* \times Q$  is the transition relation

A **configuration** of a PDA is a triple  $(q, w, \gamma)$  where

- ▶  $q$  is the current state
- ▶  $w$  is the input yet unread
- ▶  $\gamma$  is the current stack content

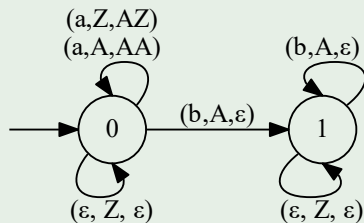
A PDA  $\mathcal{A}$  **accepts** a word  $w \in \Sigma^*$  if, starting from the configuration  $(q_0, w, Z_0)$ ,  $\mathcal{A}$  can reach the configuration  $(q, \varepsilon, \varepsilon)$  for some  $q$ .

# Example: PDA for $a^n b^n$

## Example (Automaton $\mathcal{A}$ )

$$\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, 0, Z)$$

- ▶  $Q = \{0, 1\}$
- ▶  $\Sigma = \{a, b\}$
- ▶  $\Gamma = \{A, Z\}$
- ▶  $\Delta :$



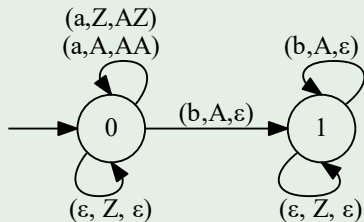
# Example: PDA for $a^n b^n$

## Example (Automaton $\mathcal{A}$ )

$$\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, 0, Z)$$

- ▶  $Q = \{0, 1\}$
- ▶  $\Sigma = \{a, b\}$
- ▶  $\Gamma = \{A, Z\}$
- ▶  $\Delta :$

0	$\varepsilon$	Z	$\rightarrow$	$\varepsilon$	0	accept empty word
0	a	Z	$\rightarrow$	AZ	0	read first a, store A
0	a	A	$\rightarrow$	AA	0	read further a, store A
0	b	A	$\rightarrow$	$\varepsilon$	1	read first b, delete A
1	b	A	$\rightarrow$	$\varepsilon$	1	read further b, delete A
1	$\varepsilon$	Z	$\rightarrow$	$\varepsilon$	1	accept if all As have been deleted





## PDA: example (2)

Process  $aabb$ :

0  $\epsilon$   $Z$   $\rightarrow$   $\epsilon$  0

0  $a$   $Z$   $\rightarrow$   $AZ$  0

0  $a$   $A$   $\rightarrow$   $AA$  0

0  $b$   $A$   $\rightarrow$   $\epsilon$  1

1  $b$   $A$   $\rightarrow$   $\epsilon$  1

1  $\epsilon$   $Z$   $\rightarrow$   $\epsilon$  1

## PDA: example (2)

Process  $aabb$ :

**1**  $(0, aabb, Z)$

$0 \ \varepsilon \ Z \rightarrow \varepsilon \ 0$

$0 \ a \ Z \rightarrow AZ \ 0$

$0 \ a \ A \rightarrow AA \ 0$

$0 \ b \ A \rightarrow \varepsilon \ 1$

$1 \ b \ A \rightarrow \varepsilon \ 1$

$1 \ \varepsilon \ Z \rightarrow \varepsilon \ 1$

## PDA: example (2)

Process  $aabb$ :

**1**  $(0, aabb, Z)$

**2**  $(0, abb, AZ)$

$0 \quad \varepsilon \quad Z \rightarrow \varepsilon \quad 0$

$0 \quad a \quad Z \rightarrow AZ \quad 0$

$0 \quad a \quad A \rightarrow AA \quad 0$

$0 \quad b \quad A \rightarrow \varepsilon \quad 1$

$1 \quad b \quad A \rightarrow \varepsilon \quad 1$

$1 \quad \varepsilon \quad Z \rightarrow \varepsilon \quad 1$

## PDA: example (2)

Process  $aabb$ :

**1**  $(0, aabb, Z)$

**2**  $(0, abb, AZ)$

**3**  $(0, bb, AAZ)$

$0 \quad \varepsilon \quad Z \rightarrow \varepsilon \quad 0$

$0 \quad a \quad Z \rightarrow AZ \quad 0$

$0 \quad a \quad A \rightarrow AA \quad 0$

$0 \quad b \quad A \rightarrow \varepsilon \quad 1$

$1 \quad b \quad A \rightarrow \varepsilon \quad 1$

$1 \quad \varepsilon \quad Z \rightarrow \varepsilon \quad 1$

## PDA: example (2)

0	$\varepsilon$	$Z$	$\rightarrow$	$\varepsilon$	0
0	$a$	$Z$	$\rightarrow$	$AZ$	0
0	$a$	$A$	$\rightarrow$	$AA$	0
0	$b$	$A$	$\rightarrow$	$\varepsilon$	1
1	$b$	$A$	$\rightarrow$	$\varepsilon$	1
1	$\varepsilon$	$Z$	$\rightarrow$	$\varepsilon$	1

Process  $aabb$ :

1 (0,  $aabb$ ,  $Z$ )

2 (0,  $abb$ ,  $AZ$ )

3 (0,  $bb$ ,  $AAZ$ )

4 (1,  $b$ ,  $AZ$ )

## PDA: example (2)

0	$\epsilon$	$Z$	$\rightarrow$	$\epsilon$	0
0	$a$	$Z$	$\rightarrow$	$AZ$	0
0	$a$	$A$	$\rightarrow$	$AA$	0
0	$b$	$A$	$\rightarrow$	$\epsilon$	1
1	$b$	$A$	$\rightarrow$	$\epsilon$	1
1	$\epsilon$	$Z$	$\rightarrow$	$\epsilon$	1

Process  $aabb$ :

**1**  $(0, aabb, Z)$

**2**  $(0, abb, AZ)$

**3**  $(0, bb, AAZ)$

**4**  $(1, b, AZ)$

**5**  $(1, \epsilon, Z)$

## PDA: example (2)

0	$\epsilon$	$Z$	$\rightarrow$	$\epsilon$	0
0	$a$	$Z$	$\rightarrow$	$AZ$	0
0	$a$	$A$	$\rightarrow$	$AA$	0
0	$b$	$A$	$\rightarrow$	$\epsilon$	1
1	$b$	$A$	$\rightarrow$	$\epsilon$	1
1	$\epsilon$	$Z$	$\rightarrow$	$\epsilon$	1

Process  $aabb$ :

**1**  $(0, aabb, Z)$

**2**  $(0, abb, AZ)$

**3**  $(0, bb, AAZ)$

**4**  $(1, b, AZ)$

**5**  $(1, \epsilon, Z)$

**6**  $(1, \epsilon, \epsilon)$

## PDA: example (2)

0	$\epsilon$	Z	$\rightarrow$	$\epsilon$	0
0	a	Z	$\rightarrow$	AZ	0
0	a	A	$\rightarrow$	AA	0
0	b	A	$\rightarrow$	$\epsilon$	1
1	b	A	$\rightarrow$	$\epsilon$	1
1	$\epsilon$	Z	$\rightarrow$	$\epsilon$	1

Process *aabb*:

1 (0, *aabb*, Z)

2 (0, *abb*, AZ)

3 (0, *bb*, AAZ)

4 (1, *b*, AZ)

5 (1,  $\epsilon$ , Z)

6 (1,  $\epsilon$ ,  $\epsilon$ )

Process *abb*:



## PDA: example (2)

0	$\epsilon$	Z	$\rightarrow$	$\epsilon$	0
0	a	Z	$\rightarrow$	AZ	0
0	a	A	$\rightarrow$	AA	0
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1	b	A	$\rightarrow$	$\epsilon$	1
1	$\epsilon$	Z	$\rightarrow$	$\epsilon$	1

Process *aabb*:

1 (0, *aabb*, Z)

2 (0, *abb*, AZ)

3 (0, *bb*, AAZ)

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5 (1,  $\epsilon$ , Z)

6 (1,  $\epsilon$ ,  $\epsilon$ )

Process *abb*:

1 (0, *abb*, Z)

## PDA: example (2)

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1	$\epsilon$	Z	$\rightarrow$	$\epsilon$	1

Process *aabb*:

**1** (0, *aabb*, Z)

**2** (0, *abb*, AZ)

**3** (0, *bb*, AAZ)

**4** (1, *b*, AZ)

**5** (1,  $\epsilon$ , Z)

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Process *abb*:

**1** (0, *abb*, Z)

**2** (0, *bb*, AZ)

## PDA: example (2)

0	$\epsilon$	Z	$\rightarrow$	$\epsilon$	0
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1	b	A	$\rightarrow$	$\epsilon$	1
1	$\epsilon$	Z	$\rightarrow$	$\epsilon$	1

Process *aabb*:

1 (0, *aabb*, Z)

2 (0, *abb*, AZ)

3 (0, *bb*, AAZ)

4 (1, *b*, AZ)

5 (1,  $\epsilon$ , Z)

6 (1,  $\epsilon$ ,  $\epsilon$ )

Process *abb*:

1 (0, *abb*, Z)

2 (0, *bb*, AZ)

3 (1, *b*, Z)

## PDA: example (2)

0	$\epsilon$	Z	$\rightarrow$	$\epsilon$	0
0	a	Z	$\rightarrow$	AZ	0
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0	b	A	$\rightarrow$	$\epsilon$	1
1	b	A	$\rightarrow$	$\epsilon$	1
1	$\epsilon$	Z	$\rightarrow$	$\epsilon$	1

Process *aabb*:

1 (0, *aabb*, Z)

2 (0, *abb*, AZ)

3 (0, *bb*, AAZ)

4 (1, *b*, AZ)

5 (1,  $\epsilon$ , Z)

6 (1,  $\epsilon$ ,  $\epsilon$ )

Process *abb*:

1 (0, *abb*, Z)

2 (0, *bb*, AZ)

3 (1, *b*, Z)

4 (1, *b*,  $\epsilon$ )

## PDA: example (2)

0	$\epsilon$	Z	$\rightarrow$	$\epsilon$	0
0	a	Z	$\rightarrow$	AZ	0
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0	b	A	$\rightarrow$	$\epsilon$	1
1	b	A	$\rightarrow$	$\epsilon$	1
1	$\epsilon$	Z	$\rightarrow$	$\epsilon$	1

Process *aabb*:

1 (0, *aabb*, Z)

2 (0, *abb*, AZ)

3 (0, *bb*, AAZ)

4 (1, *b*, AZ)

5 (1,  $\epsilon$ , Z)

6 (1,  $\epsilon$ ,  $\epsilon$ )

Process *abb*:

1 (0, *abb*, Z)

2 (0, *bb*, AZ)

3 (1, *b*, Z)

4 (1, *b*,  $\epsilon$ )

5 No rule applicable,  
input not read entirely

# PDA: important properties

- ▶  $\Gamma$  and  $\Sigma$  do not need to be disjoint
  - ▶ Not uncommon:  $\Sigma \subseteq \Gamma$
  - ▶ ... but convention:  $\Sigma$  lowercase letters,  $\Gamma$  uppercase letters
- ▶  $\varepsilon$  transitions are possible
  - ▶ ... and can modify the stack
- ▶ PDAs as defined above are **non-deterministic**
  - ▶ Deterministic PDA: For each situation there is only one applicable transition (either  $\varepsilon$  or  $c$ )
  - ▶ deterministic PDAs are strictly weaker
- ▶ We can also define PDAs with acceptance via final states, but...
  - ▶ this makes representation of PDAs more complex
  - ▶ makes proofs more difficult

Define a PDA detecting all palindromes over  $\Sigma = \{a, b\}$ , i.e. all words

$$\{w \cdot \overleftarrow{w} \mid w \in \Sigma^*\}$$

where

$$\overleftarrow{w} = a_n \dots a_1 \text{ if } w = a_1 \dots a_n$$

Can you define a deterministic automaton?

# Equivalence of PDAs and Context-Free Grammars

## Theorem

*The class of languages that can be accepted by a PDA is exactly the class of languages that can be produced by a context-free grammar.*



# Equivalence of PDAs and Context-Free Grammars

## Theorem

*The class of languages that can be accepted by a PDA is exactly the class of languages that can be produced by a context-free grammar.*

## Proof.

- ▶ For a cf. grammar  $G$ , generate a PDA  $\mathcal{A}_G$  with  $L(\mathcal{A}_G) = L(G)$ .
- ▶ For a PDA  $\mathcal{A}$ , generate a cf. grammar  $G_{\mathcal{A}}$  with  $L(G_{\mathcal{A}}) = L(\mathcal{A})$ .



# From context-free grammars to PDAs

For a grammar  $G = (N, \Sigma, P, S)$ , an equivalent PDA is:

$$\mathcal{A}_G = (\{q\}, \Sigma, \Sigma \cup N, \Delta, q, S)$$

$$\Delta = \{(q, \varepsilon, A, \gamma, q) \mid A \rightarrow \gamma \in P\} \cup \\ \{(q, a, a, \varepsilon, q) \mid a \in \Sigma\}$$

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$\mathcal{A}_G$  simulates the productions of  $G$  in the following way:

- ▶ a production rule is applied to the top stack symbol if it is an NTS
- ▶ a TS is removed from the stack if it corresponds to the next input character

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- ▶ a TS is removed from the stack if it corresponds to the next input character

Note:

- ▶  $\mathcal{A}_G$  is nondeterministic if there are several rules for one NTS.
- ▶  $\mathcal{A}_G$  only has one single state.
  - ▶ Corollary: PDAs need no states, could be written as  $(\Sigma, \Gamma, \Delta, Z_0)$ .

## From context-free grammars to PDAs: exercise

For the grammar  $G = (\{S\}, \{a, b\}, P, S)$  with

$$P = \{S \rightarrow aSa$$

$$S \rightarrow bSb$$

$$S \rightarrow \varepsilon\}$$

- ▶ create an equivalent PDA  $\mathcal{A}_G$ ,
- ▶ show how  $\mathcal{A}_G$  processes the input  $abba$ .

# From PDAs to context-free grammars

Transforming a PDA  $\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, q_0, Z_0)$  into a grammar  $G_{\mathcal{A}} = (N, \Sigma, P, S)$  is more involved:

- ▶  $N$  contains symbols  $[pZq]$ , meaning
  - ▶  $\mathcal{A}$  must go from  $p$  to  $q$  deleting  $Z$  from the stack
- ▶ for a transition  $(p, a, Z, \varepsilon, q)$  that deletes a stack symbol:
  - ▶  $\mathcal{A}$  can switch from  $p$  to  $q$  and delete  $Z$  by reading input  $a$
  - ▶ this can be expressed by a production rule  $[pZq] \rightarrow a$ .
- ▶ for transitions  $(p, a, Z, ABC, q)$  that produce stack symbols:
  - ▶ test all possible transitions for removing these symbols
  - ▶  $[pZt] \rightarrow a[qAr][rBs][sCt]$  for all states  $r, s, t$
  - ▶ intuitive meaning: in order to go from  $p$  to  $t$  and delete  $Z$ , you can
    - 1 read the input  $a$
    - 2 go into state  $q$
    - 3 find states  $r, s$  through which you can go from  $q$  to  $t$  and delete  $A, B$ , and  $C$  from the stack.

## $G_{\mathcal{A}}$ : formal definition

For  $\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, q_0, Z_0)$  we define  $G_{\mathcal{A}} = (N, \Sigma, P, S)$  as follows

- ▶  $N = \{S\} \cup \{[pZq] \mid p, q \in Q, Z \in \Gamma\}$
- ▶  $P$  contains the following rules:
  - ▶ for every  $q \in Q$ ,  $P$  contains  $\{S \rightarrow [q_0Z_0q]\}$   
meaning:  $\mathcal{A}$  has to go from  $q_0$  to any state  $q$ , deleting  $Z_0$ .
  - ▶ for each transition  $(p, a, Z, Y_1Y_2 \dots Y_n, q)$  with
    - ▶  $a \in \Sigma \cup \{\varepsilon\}$  and
    - ▶  $Z, Y_1, Y_2 \dots Y_n \in \Gamma$ ,

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  - ▶ for each transition  $(p, a, Z, Y_1Y_2 \dots Y_n, q)$  with
    - ▶  $a \in \Sigma \cup \{\varepsilon\}$  and
    - ▶  $Z, Y_1, Y_2 \dots Y_n \in \Gamma$ ,

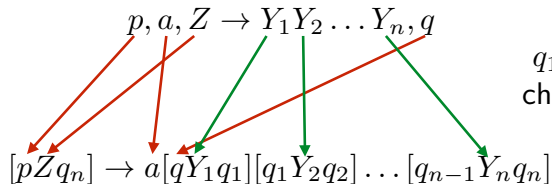
$P$  contains rules

$$[pZq_n] \rightarrow a[qY_1q_1][q_1Y_2q_2] \dots [q_{n-1}Y_nq_n]$$

for all possible combinations of states  $q_1, q_2, \dots, q_n \in Q$ .



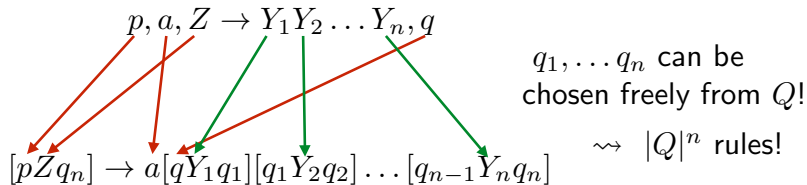
# PDA to grammar illustrated



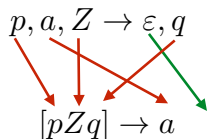
$q_1, \dots, q_n$  can be  
chosen freely from  $Q$ !

$\rightsquigarrow |Q|^n$  rules!

# PDA to grammar illustrated

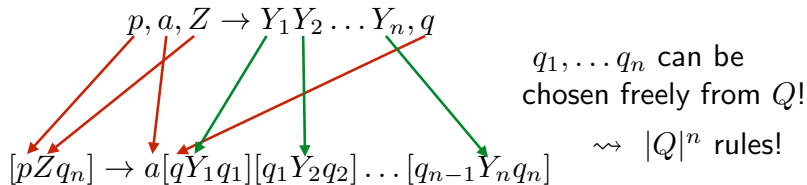


Special case:  $n = 0$

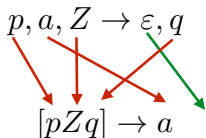


$\rightsquigarrow 1$  rule!

# PDA to grammar illustrated

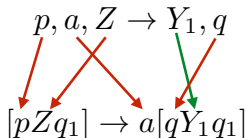


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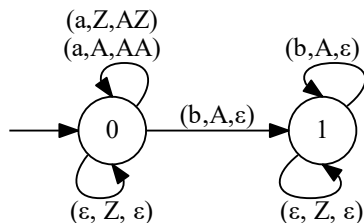


$\rightsquigarrow |Q|$  rules!

# Exercise: transformation of PDA into grammar

$\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, 0, Z)$

- ▶  $Q = \{0, 1\}$
- ▶  $\Sigma = \{a, b\}$
- ▶  $\Gamma = \{A, Z\}$
- ▶  $\Delta :$



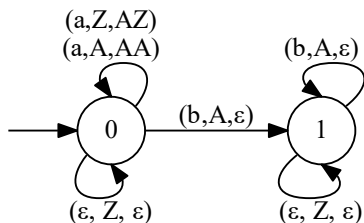
0	$\epsilon$	Z	$\rightarrow$	$\epsilon$	0
0	a	Z	$\rightarrow$	AZ	0
0	a	A	$\rightarrow$	AA	0
0	b	A	$\rightarrow$	$\epsilon$	1
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1	$\epsilon$	Z	$\rightarrow$	$\epsilon$	1

- ▶ Transform  $\mathcal{A}$  into a grammar  $G_{\mathcal{A}}$  (and reduce  $G_{\mathcal{A}}$ ).
- ▶ Show how  $G_{\mathcal{A}}$  produces the words  $\epsilon$ ,  $ab$ , and  $aabb$ .

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## Bonus Exercises/Homework

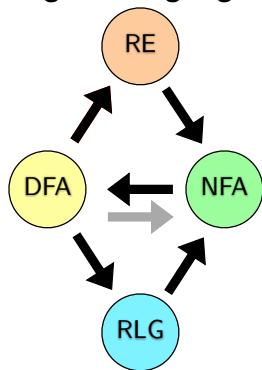
- ▶ Assume  $\Sigma = \{a, b\}$ .
  - ▶ Find a PDA  $A_1$  that accepts  $L_1 = \{w \in \Sigma^* \mid |w|_a = |w|_b\}$ .
  - ▶ Give an accepting run of  $A_1$  on *abbbaa*.
- ▶ Assume  $\Sigma = \{a, b\}$ .
  - ▶ Find a PDA  $A_2$  that accepts  $L_2 = \{w \in \Sigma^* \mid |w|_a < |w|_b\}$ .
  - ▶ Give an accepting run of  $A_2$  on *bbbaaaabb*.
- ▶ Assume  $\Sigma = \{a, b, c\}$ .
  - ▶ Find a PDA  $A_3$  that accepts  $L_3 = \{w \in \Sigma^* \mid |w| \text{ is odd and } w[(|w| + 1)/2] = a\}$  (the middle symbol is an *a*)
  - ▶ Give an accepting run of  $A_3$  on *cccaabb*.
- ▶ Assume  $\Sigma = \{a, b, c\}$ .
  - ▶ Find a PDA  $A_4$  that accepts  $L_4 = \{a^n b^m c^o \mid n, m, o \in \mathbb{N}, n = m + o\}$
  - ▶ Give an accepting run of  $A_4$  on *aacc*.

# Bonus Exercises/Homework

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  - ▶ Find a PDA  $A_1$  that accepts  $L_1 = \{w \in \Sigma^* \mid |w|_a = |w|_b\}$ .
  - ▶ Give an accepting run of  $A_1$  on *abbbbaa*.
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  - ▶ Find a PDA  $A_2$  that accepts  $L_2 = \{w \in \Sigma^* \mid |w|_a < |w|_b\}$ .
  - ▶ Give an accepting run of  $A_2$  on *bbbaaaabb*.
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  - ▶ Give an accepting run of  $A_4$  on *aacc*.

# Comparison: Regular vs. context-free languages

## Regular languages



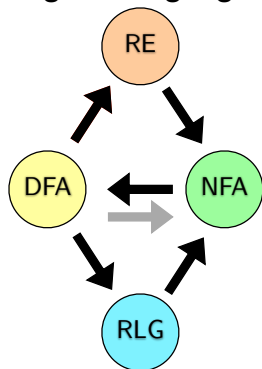
## Context-free languages





# Comparison: Regular vs. context-free languages

## Regular languages



## Context-free languages



For both language classes there are different but equivalent formal descriptions, supporting different arguments about the classes!

# Outline

Introduction

Regular Languages and Finite Automata

Scanners and Flex

**Formal Grammars and Context-Free Languages**

Formal Grammars

The Chomsky Hierarchy

Right-linear Grammars

Context-free Grammars

Push-Down Automata

**Properties of Context-free Languages**

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Lecture-specific material

Bonus Exercises

Selected Solutions

## Theorem (Existence of non-context-free languages)

*Let  $\Sigma$  be an alphabet. There are languages  $L \subseteq \Sigma^*$  such that there is no context-free grammar  $G$  with  $L = L(G)$ .*

- ▶ There are languages that are not context-free
  - ▶ e.g.  $\{a^n b^n c^n \mid n \in \mathbb{N}\}$
  - ▶ ... but how do we show this?
- ▶ For regular languages: Pumping lemma
  - ▶ Finite automata must loop on long words
  - ▶ Loops can be repeated
  - ▶ Hence: A language that cannot be pumped is not regular
- ▶ For context-free languages?

# Pumping-lemma for context-free languages

## Idea:

- ▶ If a context-free grammar  $G$  can produce words of arbitrary length, there is at least one **repeated NTS** in the derivation
  - ▶ If there are  $n$  rules in the grammar, at least one rule has to be used more than once in a derivation of length  $n + 1$
  - ▶ Slightly stronger arguments (based on  $|N|$  instead of  $|P|$  are possible)
- ▶ Hence there is a derivation  $A \Longrightarrow^* vAx$  for  $v, x \in (\Sigma \cup \Gamma)^*$ 
  - ▶ ... and we can repeat this part of the derivation  
 $A \Longrightarrow^* vAx \Longrightarrow^* vvAxx \Longrightarrow^* vvvAxxx$
  - ▶ If  $G$  has no chain rules, at least one of  $v, x$  is non-empty

# Pumping Lemma I vs. Pumping Lemma II:

- ▶ PL I (regular languages)
  - ▶ Argument based on accepting automaton
  - ▶ Finite automaton must loop on sufficiently long word
  - ▶ One non-empty segment can be pumped
- ▶ PL II (context-free languages)
  - ▶ Argument based on generating grammar
  - ▶ At least one non-terminal must repeat in sufficiently long derivation
  - ▶ Two segments around this NTS can be pumped in parallel, at least one of which is non-empty

# Pumping Lemma II

## Theorem (Pumping-Lemma for context-free languages)

Let  $L$  be a language generated by a context-free grammar  $G_L = (N, \Sigma, P, S)$  without contracting rules or chain rules. Let  $m = |N|$ ,  $r$  be the maximum length of the rhs of a rule in  $P$ , and  $k = r \cdot (m + 1)$ .

Then:

For every  $s \in L$  with  $|s| > k$  there exists a segmentation  $u \cdot v \cdot w \cdot x \cdot y = s$  such that

- 1  $vx \neq \varepsilon$
- 2  $|vwx| \leq k$
- 3  $u \cdot v^h \cdot w \cdot x^h \cdot y \in L$  for every  $h \in \mathbb{N}$ .

# Pumping Lemma II

## Theorem (Pumping-Lemma for context-free languages)

*Let  $L$  be a context-free language. Then there exists  $k \in \mathbb{N}$  with the following property:*

*For every  $s \in L$  with  $|s| > k$  there exists a segmentation  $u \cdot v \cdot w \cdot x \cdot y = s$  such that*

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- 3  $u \cdot v^h \cdot w \cdot x^h \cdot y \in L$  for every  $h \in \mathbb{N}$ .

- ▶ Holds for  $\{a^n b^n\}$ , but not for  $\{a^n b^n c^n\}$ .
- ▶  $\{a^n b^n c^n\}$  is **not context-free**, but context-sensitive, as we have seen before.



## Group Exercise: $a^n b^n c^n$

Use the Pumping Lemma II to show that  $L = \{a^n b^n c^n \mid n \in \mathbb{N}\}$  is not context-free.

# Closure properties

## Theorem (Closure under $\cup, \cdot, *$ )

*The class of context-free languages is closed under union, concatenation, and Kleene star.*

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For context-free grammars

$$G_1 = (N_1, \Sigma, P_1, S_1) \quad \text{and} \quad G_2 = (N_2, \Sigma, P_2, S_2)$$

with  $N_1 \cap N_2 = \{\}$  (rename NTSs if needed), let  $S$  be a new start symbol.

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For context-free grammars

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with  $N_1 \cap N_2 = \{\}$  (rename NTSs if needed), let  $S$  be a new start symbol.

- ▶ for  $L(G_1) \cup L(G_2)$ , add productions  $S \rightarrow S_1, S \rightarrow S_2$ .
- ▶ for  $L(G_1) \cdot L(G_2)$ , add production  $S \rightarrow S_1 S_2$ .
- ▶ for  $L(G_1)^*$ , add productions  $S \rightarrow \varepsilon, S \rightarrow S_1 S$ .

### Theorem (Closure under $\cap$ )

*Context-free languages are not closed under intersection.*

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*Context-free languages are not closed under intersection.*

Otherwise,  $\{a^n b^n c^n\}$  would be context-free:

- ▶  $\{a^n b^n c^m\}$  is context-free
- ▶  $\{a^m b^n c^n\}$  is context-free
- ▶  $\{a^n b^n c^n\} = \{a^n b^n c^m\} \cap \{a^m b^n c^n\}$

## Exercise: closure properties

- 1 Define context-free grammars for  $L_1 = \{a^n b^n c^m \mid n, m \geq 0\}$  and  $L_2 = \{a^m b^n c^n \mid n, m \geq 0\}$ .
- 2 Use  $L_1, L_2$  and the known closure properties to show that context-free languages are not closed under complement.

# Decision problems: word problem

## Theorem (Word problem for cf. languages)

*For a word  $w$  and a context-free grammar  $G$ , it is **decidable** whether  $w \in L(G)$  holds.*



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*For a word  $w$  and a context-free grammar  $G$ , it is **decidable** whether  $w \in L(G)$  holds.*

## Proof.

The CYK algorithm decides the word problem. □

## Decision problems: emptiness problem

Theorem (Emptiness problem for cf. languages)

*For a context-free grammar  $G$ , it is **decidable** if  $L(G) = \{\}$  holds.*

# Decision problems: emptiness problem

## Theorem (Emptiness problem for cf. languages)

For a context-free grammar  $G$ , it is *decidable* if  $L(G) = \{\}$  holds.

## Proof.

Let  $G = (N, \Sigma, P, S)$ .

Iteratively compute *productive* NTSs, i.e. symbols that produce terminal words as follows:

- 1 let  $Z = \Sigma$
- 2 add all symbols  $A$  to  $Z$  for which there is a rule  $A \rightarrow \beta$  with  $\beta \in Z^*$
- 3 repeat step 2 until no further symbols can be added
- 4  $L(G) = \{\}$  iff  $S \notin Z$ .



## Decision problems: equivalence problem

### Theorem (Equivalence problem for cf. languages)

*For context-free grammars  $G_1, G_2$ , it is **undecidable** if  $L(G_1) = L(G_2)$  holds.*

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This follows from undecidability of Post's Correspondence Problem.

- ▶ A PCP can be encoded in grammars such that the PCP has a solution if and only if the two grammars are equivalent
- ▶ Since the PCP is undecidable, so is the equivalence problem
- ▶ Detail? Who needs details?

# Summary: context-free languages

- ▶ characterised by
  - ▶ context-free grammars
  - ▶ pushdown automata
- ▶ closure properties
  - ▶ closed under  $\cup, *, \cdot$
  - ▶ not closed under  $\cap, \bar{\phantom{x}}$
- ▶ decision problems
  - ▶ decidable:  $w \in L(G), L(G) = \{\}$  (Chomsky NF, CYK algorithm)
  - ▶ undecidable:  $L(G_1) = L(G_2)$
- ▶ can describe **nested dependencies**
  - ▶ structure of programming languages
  - ▶ natural language processing
- ▶ in compilers, these features are used by **parsers** (next chapter)

# Outline

Introduction

Regular Languages and Finite Automata

Scanners and Flex

Formal Grammars and Context-Free Languages

**Parsers and Bison**

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# Parsing: Motivation

Formal grammars describe formal languages

- ▶ A grammar has a set of rules
- ▶ Rules replace words with words
- ▶ A word that can be derived from the start symbol belongs to the language of the grammar



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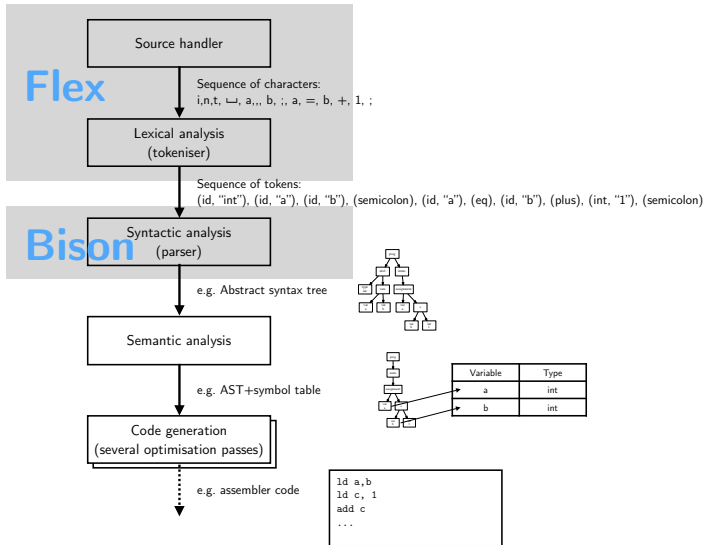
**In the concrete case of programming languages,  
“words of the language” are syntactically correct programs!**

- ▶ Most programming languages are described by context-free grammars (with extra “semantic” constraints)
- ▶ Grammars **generate** languages
- ▶ PDAs and e.g. CYK-Parsing **recognize** words
- ▶ For compiler we need to ...
  - ▶ identify correct programs
  - ▶ and understand their structure!

# Lexing and Parsing

- ▶ **Lexer: Breaks programs into tokens**
  - ▶ Smallest parts with semantic meaning
  - ▶ Can be recognized by regular languages/patterns
  - ▶ Example: `1`, `2`, `53` are all Integers
  - ▶ Example: `i`, `handle`, `stream` are all Identifiers
  - ▶ Example: `>`, `>=`, `*` are all individual operators
- ▶ **Parser: Recognizes program structure**
  - ▶ Language described by a grammar that has token types as terminals, not individual characters
  - ▶ Parser builds **parse tree**

# Automatisation with Bison



- ▶ Yacc - Yet Another Compiler Compiler
  - ▶ Originally written  $\approx 1971$  by Stephen C. Johnson at AT&T
  - ▶ LALR parser generator
  - ▶ Translates grammar into syntax analyser



# YACC/Bison

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- ▶ GNU Bison
  - ▶ Written by Robert Corbett in 1988
  - ▶ Yacc-compatibility by Richard Stallman
  - ▶ Output languages now C, C++, Java
- ▶ Yacc, Bison, BYacc, . . . mostly compatible (POSIX P1003.2)

# Yacc/Bison Background

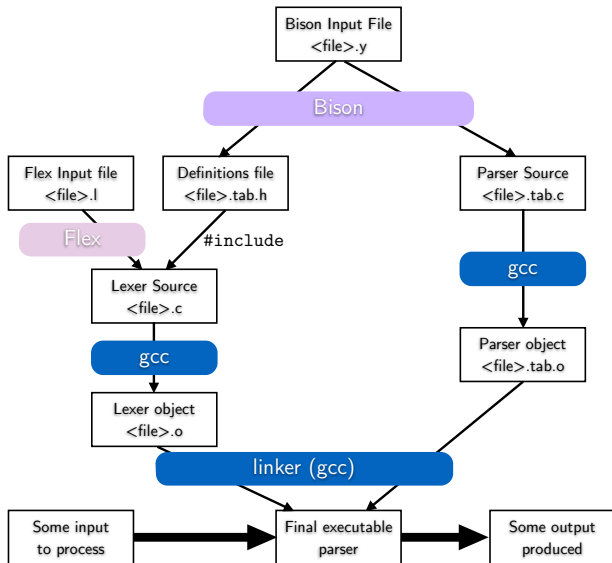
- ▶ By default, Bison constructs a **1 token Look-Ahead Left-to-right Rightmost-derivation** or LALR(1) parser
  - ▶ Input tokens are processed **left-to-right**
  - ▶ Shift-reduce parser:
    - ▶ **Stack** holds tokens (terminals) and non-terminals
    - ▶ Tokens are **shifted** from input to stack. If the top of the stack contains symbols that represent the right hand side (RHS) of a grammar rule, the content is **reduced** to the LHS
    - ▶ Since input is reduced left-to-right, this corresponds to a **rightmost** derivation
    - ▶ Ambiguities are solved via look-ahead and special rules
    - ▶ If input can be reduced to start symbol, success!
    - ▶ Error otherwise
- ▶ LALR(1) is efficient in time and memory
  - ▶ Can parse “all reasonable languages”
  - ▶ For unreasonable languages, Bison (but not Yacc) can also construct **GLR** (General LR) parsers
    - ▶ Try all possibilities with back-tracking
    - ▶ Corresponds to the **non-determinism** of stack machines

# Yacc/Bison Overview

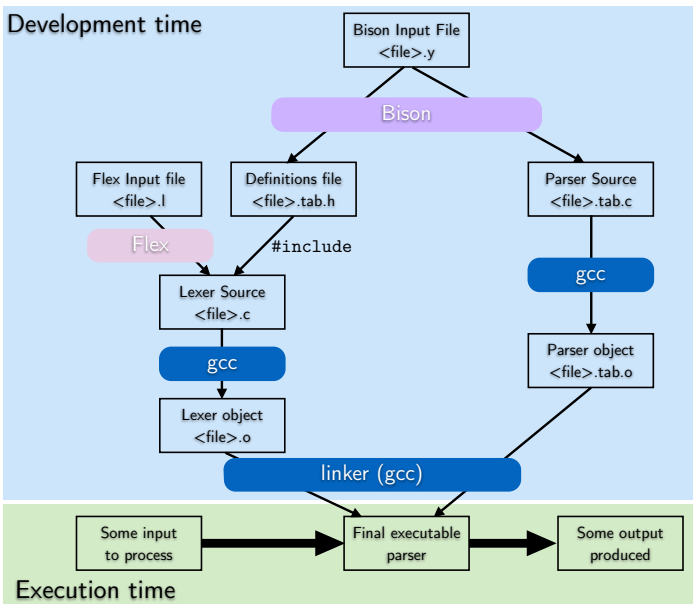
- ▶ Bison reads a specification file and converts it into (C) code of a parser
- ▶ Specification file: Declarations, grammar rules with actions, support code
  - ▶ Declarations: C declarations and data model, token names, associated values, includes
  - ▶ Grammar rules: Non-terminal with alternatives, **action** associated with each alternative
  - ▶ Support code: e.g. `main()` function, error handling...
  - ▶ Syntax similar to (F)lex
    - ▶ Sections separated by `%%`
    - ▶ Special commands start with `%`
- ▶ Bison generates function `yyparse()`
- ▶ Bison needs function `yylex()`
  - ▶ Usually provided via (F)lex



# Yacc/Bison workflow



# Yacc/Bison workflow



# Example task: Desk calculator

- ▶ Desk calculator
  - ▶ Reads algebraic expressions and assignments
  - ▶ Prints result of expressions
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```

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> RegVal: 87.820500

R9+R10
> 114.820500

...
```

# Abstract grammar for desk calculator (partial)

$$G_{DC} = (N, \Sigma, P, S)$$

- ▶  $\Sigma = \{\text{PLUS, MULT, ASSIGN, OPENPAR, CLOSEPAR, REGISTER, FLOAT, ...}\}$ 
  - ▶ Some terminals are single characters  
(+, =, ...)
  - ▶ Others are complex  
R10, 1.3e7
  - ▶ Terminals (“tokens”) are generated by the lexer
- ▶  $N = \{\text{stmt, assign, expr, term, factor, ...}\}$



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▶  $P:$   
stmt  $\rightarrow$  assign  
|  
expr

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stmt	→	assign
		expr
assign	→	REGISTER ASSIGN expr

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assign	→	REGISTER ASSIGN expr
expr	→	expr PLUS term
		term

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stmt	→ assign   expr
assign	→ REGISTER ASSIGN expr
expr	→ expr PLUS term   term
term	→ term MULT factor   factor

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		expr
assign	→	REGISTER ASSIGN expr
expr	→	expr PLUS term
		term
term	→	term MULT factor
		factor
factor	→	REGISTER
		FLOAT
		OPENPAR expr CLOSEPAR

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		expr
assign	→	REGISTER ASSIGN expr
expr	→	expr PLUS term
		term
term	→	term MULT factor
		factor
factor	→	REGISTER
		FLOAT
		OPENPAR expr CLOSEPAR

- ▶  $S$ :
- ▶ For a single statement,  $S = \text{stmt}$
  - ▶ In practice, we need to handle sequences of statements and empty input lines (not reflected in the grammar)

# Parsing statements (1)

- ▶ Example string:  $R10 = (4.5+3*7)$
- ▶ Tokenized: REGISTER ASSIGN OPENPAR FLOAT PLUS  
FLOAT MULT FLOAT CLOSEPAR
  - ▶ In the following abbreviated R, A, O, F, P, F, M, F, C
- ▶ Parsing state:
  - ▶ Unread input (left column)
  - ▶ Current stack (middle column, top element on right)
  - ▶ How state was reached (right column)



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Input	Stack	Comment
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Input	Stack	Comment
R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack

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O F P F M F C	R A	Shift A to stack

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F P F M F C	R A O	Shift O to stack

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F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack

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A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F

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F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor

# Parsing statements (1)

- ▶ Example string:  $R10 = (4.5+3*7)$
- ▶ Tokenized: REGISTER ASSIGN OPENPAR FLOAT PLUS  
FLOAT MULT FLOAT CLOSEPAR
  - ▶ In the following abbreviated R, A, O, F, P, F, M, F, C
- ▶ Parsing state:
  - ▶ Unread input (left column)
  - ▶ Current stack (middle column, top element on right)
  - ▶ How state was reached (right column)
- ▶ Parsing:

Input	Stack	Comment
R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor

...



## Parsing statements (2)

R A O F P F M F C

A O F P F M F C

O F P F M F C

F P F M F C

P F M F C

P F M F C

P F M F C

R

R A

R A O

R A O F

R A O factor

R A O term

Start

Shift R to stack

Shift A to stack

Shift O to stack

Shift F to stack

Reduce F

Reduce factor

## Parsing statements (2)

R A O F P F M F C	
A O F P F M F C	R
O F P F M F C	R A
F P F M F C	R A O
P F M F C	R A O F
P F M F C	R A O factor
P F M F C	R A O term
P F M F C	R A O expr

Start  
Shift R to stack  
Shift A to stack  
Shift O to stack  
Shift F to stack  
Reduce F  
Reduce factor  
Reduce term

## Parsing statements (2)

R A O F P F M F C	
A O F P F M F C	R
O F P F M F C	R A
F P F M F C	R A O
P F M F C	R A O F
P F M F C	R A O factor
P F M F C	R A O term
P F M F C	R A O expr
F M F C	R A O expr P

Start  
Shift R to stack  
Shift A to stack  
Shift O to stack  
Shift F to stack  
Reduce F  
Reduce factor  
LA! Reduce term  
Shift P

## Parsing statements (2)

R A O F P F M F C

A O F P F M F C

O F P F M F C

F P F M F C

P F M F C

P F M F C

P F M F C

P F M F C

F M F C

M F C

R

R A

R A O

R A O F

R A O factor

R A O term

R A O expr

R A O expr P

R A O expr P F

Start

Shift R to stack

Shift A to stack

Shift O to stack

Shift F to stack

Reduce F

Reduce factor

LA! Reduce term

Shift P

Shift F

## Parsing statements (2)

R A O F P F M F C	
A O F P F M F C	R
O F P F M F C	R A
F P F M F C	R A O
P F M F C	R A O F
P F M F C	R A O factor
P F M F C	R A O term
P F M F C	R A O expr
F M F C	R A O expr P
M F C	R A O expr P F
M F C	R A O expr P factor

Start  
Shift R to stack  
Shift A to stack  
Shift O to stack  
Shift F to stack  
Reduce F  
Reduce factor  
LA! Reduce term  
Shift P  
Shift F  
Reduce F

## Parsing statements (2)

R A O F P F M F C

A O F P F M F C

O F P F M F C

F P F M F C

P F M F C

P F M F C

P F M F C

P F M F C

F M F C

M F C

M F C

M F C

R

R A

R A O

R A O F

R A O factor

R A O term

R A O expr

R A O expr P

R A O expr P F

R A O expr P factor

R A O expr P term

Start

Shift R to stack

Shift A to stack

Shift O to stack

Shift F to stack

Reduce F

Reduce factor

LA! Reduce term

Shift P

Shift F

Reduce F

Reduce factor

## Parsing statements (2)

R A O F P F M F C	
A O F P F M F C	R
O F P F M F C	R A
F P F M F C	R A O
P F M F C	R A O F
P F M F C	R A O factor
P F M F C	R A O term
P F M F C	R A O expr
F M F C	R A O expr P
M F C	R A O expr P F
M F C	R A O expr P factor
M F C	R A O expr P term
F C	R A O expr P term M

Start  
Shift R to stack  
Shift A to stack  
Shift O to stack  
Shift F to stack  
Reduce F  
Reduce factor  
LA! Reduce term  
Shift P  
Shift F  
Reduce F  
Reduce factor  
LA! Shift M

## Parsing statements (2)

R A O F P F M F C	
A O F P F M F C	R
O F P F M F C	R A
F P F M F C	R A O
P F M F C	R A O F
P F M F C	R A O factor
P F M F C	R A O term
P F M F C	R A O expr
F M F C	R A O expr P
M F C	R A O expr P F
M F C	R A O expr P factor
M F C	R A O expr P term
F C	R A O expr P term M
C	R A O expr P term M F

Start  
Shift R to stack  
Shift A to stack  
Shift O to stack  
Shift F to stack  
Reduce F  
Reduce factor  
LA! Reduce term  
Shift P  
Shift F  
Reduce F  
Reduce factor  
LA! Shift M  
Shift F



## Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F

## Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf

## Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt

## Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt
	R A O expr C	Shift C

## Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt
	R A O expr C	Shift C
	R A factor	Reduce OeC

## Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt
	R A O expr C	Shift C
	R A factor	Reduce OeC
	R A term	Reduce factor

## Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt
	R A O expr C	Shift C
	R A factor	Reduce OeC
	R A term	Reduce factor
	R A expr	Reduce term

## Parsing statements (2)

R A O F P F M F C		Start
A O F P F M F C	R	Shift R to stack
O F P F M F C	R A	Shift A to stack
F P F M F C	R A O	Shift O to stack
P F M F C	R A O F	Shift F to stack
P F M F C	R A O factor	Reduce F
P F M F C	R A O term	Reduce factor
P F M F C	R A O expr	LA! Reduce term
F M F C	R A O expr P	Shift P
M F C	R A O expr P F	Shift F
M F C	R A O expr P factor	Reduce F
M F C	R A O expr P term	Reduce factor
F C	R A O expr P term M	LA! Shift M
C	R A O expr P term M F	Shift F
C	R A O expr P term M factor	Reduce F
C	R A O expr P term	Reduce tMf
C	R A O expr	Reduce ePt
	R A O expr C	Shift C
	R A factor	Reduce OeC
	R A term	Reduce factor
	R A expr	Reduce term
	assign	Reduce RAe



## Parsing statements (2)

R A O F P F M F C

A O F P F M F C

O F P F M F C

F P F M F C

P F M F C

P F M F C

P F M F C

P F M F C

F M F C

M F C

M F C

M F C

F C

C

C

C

C

R

R A

R A O

R A O F

R A O factor

R A O term

R A O expr

R A O expr P

R A O expr P F

R A O expr P factor

R A O expr P term

R A O expr P term M

R A O expr P term M F

R A O expr P term M factor

R A O expr P term

R A O expr

R A O expr C

R A factor

R A term

R A expr

assign

stmt

Start

Shift R to stack

Shift A to stack

Shift O to stack

Shift F to stack

Reduce F

Reduce factor

LA! Reduce term

Shift P

Shift F

Reduce F

Reduce factor

LA! Shift M

Shift F

Reduce F

Reduce tMf

Reduce ePt

Shift C

Reduce OeC

Reduce factor

Reduce term

Reduce RAe

Reduce assign

# Lexer interface

- ▶ Bison parser requires `yylex()` function
- ▶ `yylex()` returns **token**
  - ▶ Token text is defined by regular expression pattern
  - ▶ Tokens are encoded as integers
  - ▶ Symbolic names for tokens are defined by Bison in generated header file
    - ▶ By convention: Token names are all `CAPITALS`
- ▶ `yylex()` provides optional **semantic value** of token
  - ▶ Stored in global variable `yylval`
  - ▶ Type of `yylval` defined by Bison in generated header file
    - ▶ Default is `int`
    - ▶ For more complex situations often a `union`
    - ▶ For our example: Union of `double` (for floating point values) and `integer` (for register numbers)

# Lexer for desk calculator (1)

```
/*  
Lexer for a minimal "scientific" calculator.  
  
Copyright 2014 by Stephan Schulz, schulz@eprover.org.  
  
This code is released under the GNU General Public Licence  
Version 2.  
*/  
  
%option noyywrap  
  
%{  
    #include "scicalcparse.tab.h"  
%}
```

## Lexer for desk calculator (2)

```
DIGIT      [0-9]
INT        {DIGIT}+
PLAINFLOAT {INT} | {INT} [.] | {INT} [.] {INT} | [.] {INT}
EXP        [eE] (\+|-)?{INT}
NUMBER     {PLAINFLOAT}{EXP}?
REG        R{DIGIT}{DIGIT}?
```

```
%%
```

```
"*" {return MULT;}
"+" {return PLUS;}
"=" {return ASSIGN;}
"(" {return OPENPAR;}
")" {return CLOSEPAR;}
\n  {return NEWLINE;}
```

## Lexer for desk calculator (3)

```
{REG}    {
    yylval.regno = atoi(yytext+1);
    return REGISTER;
}

{NUMBER} {
    yylval.val = atof(yytext);
    return FLOAT;
}

[ ] { /* Skip whitespace*/ }

/* Everything else is an invalid character. */
.    { return ERROR;}

%%
```

# Data model of desk calculator

- ▶ Desk calculator has simple state
  - ▶ 100 floating point registers
  - ▶ R0-R99

- ▶ Represented in C as array of doubles:

```
#define MAXREGS 100
```

```
double regfile[MAXREGS];
```

- ▶ Needs to be initialized in support code

## Bison code for desk calculator: Declarations

```
%{
#include <stdio.h>

#define MAXREGS 100

double regfile[MAXREGS];

extern int yyerror(char* err);
extern int yylex(void);
}%

%union {
double val;
int regno;
}
```

# Bison code for desk calculator: Tokens

```
%start stmtseq
```

```
%left PLUS
```

```
%left MULT
```

```
%token ASSIGN
```

```
%token OPENPAR
```

```
%token CLOSEPAR
```

```
%token NEWLINE
```

```
%token REGISTER
```

```
%token FLOAT
```

```
%token ERROR
```

```
%%
```



# Actions in Bison

- ▶ Bison is based on syntax rules with associated actions
  - ▶ Whenever a **reduce** is performed, the action associated with the rule is executed
- ▶ Actions can be arbitrary C code
- ▶ Frequent: **semantic actions**
  - ▶ The action sets a **semantic value** based on the semantic value of the symbols reduced by the rule
  - ▶ For terminal symbols: Semantic value is `yyval` from Flex
  - ▶ Semantic actions have “historically valuable” syntax
    - ▶ Value of reduced symbol: `$$`
    - ▶ Value of first symbol in syntax rule body: `$1`
    - ▶ Value of second symbol in syntax rule body: `$2`
    - ▶ ...
    - ▶ Access to named components: `$(val>1)`

## Bison code for desk calculator: Grammar (1)

```
stmtseq: /* Empty */
        | NEWLINE stmtseq      {}
        | stmt NEWLINE stmtseq {}
        | error NEWLINE stmtseq {}; /* After an error,
start afresh */
```

- ▶ Head: sequence of statements
- ▶ First body line: Skip empty lines
- ▶ Second body line: separate current statement from rest
- ▶ Third body line: After parse error, start again with new line

## Bison code for desk calculator: Grammar (2)

```
stmt: assign {printf("> RegVal: %f\n", $<val>1);}
      |expr   {printf("> %f\n", $<val>1);};

assign: REGISTER ASSIGN expr {regfile[$<regno>1] = $<val>3;
      $<val>$ = $<val>3;} ;

expr: expr PLUS term {$<val>$ = $<val>1 + $<val>3;}
      | term {$<val>$ = $<val>1};};

term: term MULT factor {$<val>$ = $<val>1 * $<val>3;}
      | factor {$<val>$ = $<val>1};};

factor: REGISTER {$<val>$ = regfile[$<regno>1];}
        | FLOAT {$<val>$ = $<val>1;}
        | OPENPAR expr CLOSEPAR {$<val>$ = $<val>2};};
```

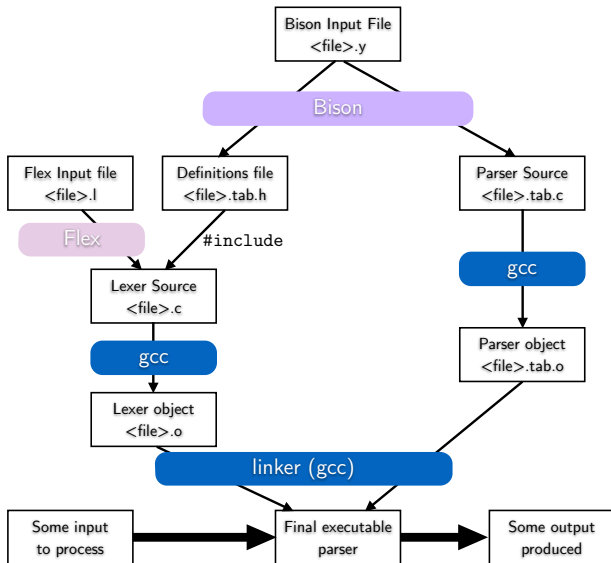
## Bison code for desk calculator: Support code

```
int yyerror(char* err)
{
    printf("Error: %s\n", err);
    return 0;
}

int main (int argc, char* argv[])
{
    int i;

    for(i=0; i<MAXREGS; i++)
    {
        regfile[i] = 0.0;
    }
    return yyparse();
}
```

# Bison workflow and dependencies



# Building the calculator

## 1 Generate parser C code and include file for lexer

- ▶ `bison -d scicalcparse.y`
- ▶ Generates `scicalcparse.tab.c` and `scicalcparse.tab.h`

## 2 Generate lexer C code

- ▶ `flex -t scicalclex.l > scicalclex.c`

## 3 Compile lexer

- ▶ `gcc -c -o scicalclex.o scicalclex.c`

## 4 Compile parser and support code

- ▶ `gcc -c -o scicalcparse.tab.o scicalcparse.tab.c`

## 5 Link everything

- ▶ `gcc scicalclex.o scicalcparse.tab.o -o scicalc`

## 6 Fun!

- ▶ `./scicalc`

# Exercise: calculator

## ▶ Exercise 1:

- ▶ Go to `http://wwwlehre.dhbw-stuttgart.de/~sschulz/fla2020.html`
- ▶ Download `scicalcparse.y` and `scicalclex.l`
- ▶ Build the calculator
- ▶ Run and test the calculator

## ▶ Exercise 2:

- ▶ Add support for division and subtraction `/`, `-`
- ▶ Add support for unary minus (the negation operator `-`)

## Definition (derivation)

For a grammar  $G$ , a **derivation** of a word  $w_n$  in  $L(G)$  is a sequence  $S \Rightarrow w_1 \Rightarrow \dots \Rightarrow w_n$  where  $S$  is the start symbol, and each  $w_i$  is generated from its predecessor by application of a production of the grammar



# Example: derivation

## Example (well-formed expressions over $x, +, *, (, )$ )

Let  $G_E = (N, \Sigma, P, E)$  with

- ▶  $N = \{E\}$
- ▶  $\Sigma = \{(, ), +, *, x\}$
- ▶  $P$ :
  - 1  $E \rightarrow x$
  - 2  $E \rightarrow (E)$
  - 3  $E \rightarrow E + E$
  - 4  $E \rightarrow E * E$

# Example: derivation

## Example (well-formed expressions over $x, +, *, (, )$ )

Let  $G_E = (N, \Sigma, P, E)$  with

- ▶  $N = \{E\}$
- ▶  $\Sigma = \{(, ), +, *, x\}$
- ▶ P:

- 1  $E \rightarrow x$
- 2  $E \rightarrow (E)$
- 3  $E \rightarrow E + E$
- 4  $E \rightarrow E * E$

The following is a derivation of  $x + x + x * x$  (with the replaced symbol printed bold):

**E**

# Example: derivation

## Example (well-formed expressions over $x, +, *, (, )$ )

Let  $G_E = (N, \Sigma, P, E)$  with

- ▶  $N = \{E\}$
- ▶  $\Sigma = \{(, ), +, *, x\}$
- ▶ P:

- 1  $E \rightarrow x$
- 2  $E \rightarrow (E)$
- 3  $E \rightarrow E + E$
- 4  $E \rightarrow E * E$

The following is a derivation of  $x + x + x * x$  (with the replaced symbol printed bold):

$$\begin{array}{l} E \\ \Rightarrow \mathbf{E} + E \end{array}$$

# Example: derivation

## Example (well-formed expressions over $x, +, *, (, )$ )

Let  $G_E = (N, \Sigma, P, E)$  with

- ▶  $N = \{E\}$
- ▶  $\Sigma = \{(, ), +, *, x\}$
- ▶  $P$ :

**1**  $E \rightarrow x$

**2**  $E \rightarrow (E)$

**3**  $E \rightarrow E + E$

**4**  $E \rightarrow E * E$

The following is a derivation of  $x + x + x * x$  (with the replaced symbol printed bold):

$$\begin{aligned} & \mathbf{E} \\ \Rightarrow & \mathbf{E} + E \\ \Rightarrow & E + E + \mathbf{E} \end{aligned}$$

# Example: derivation

## Example (well-formed expressions over $x, +, *, (, )$ )

Let  $G_E = (N, \Sigma, P, E)$  with

- ▶  $N = \{E\}$
- ▶  $\Sigma = \{(, ), +, *, x\}$
- ▶  $P$ :

**1**  $E \rightarrow x$

**2**  $E \rightarrow (E)$

**3**  $E \rightarrow E + E$

**4**  $E \rightarrow E * E$

The following is a derivation of  $x + x + x * x$  (with the replaced symbol printed bold):

$$\begin{aligned} & \mathbf{E} \\ \Rightarrow & \mathbf{E} + E \\ \Rightarrow & E + E + \mathbf{E} \\ \Rightarrow & \mathbf{E} + E + E * E \end{aligned}$$

# Example: derivation

## Example (well-formed expressions over $x, +, *, (, )$ )

Let  $G_E = (N, \Sigma, P, E)$  with

- ▶  $N = \{E\}$
- ▶  $\Sigma = \{(, ), +, *, x\}$
- ▶ P:

- 1  $E \rightarrow x$
- 2  $E \rightarrow (E)$
- 3  $E \rightarrow E + E$
- 4  $E \rightarrow E * E$

The following is a derivation of  $x + x + x * x$  (with the replaced symbol printed bold):

$$\begin{aligned} & \mathbf{E} \\ \Rightarrow & \mathbf{E} + E \\ \Rightarrow & E + E + \mathbf{E} \\ \Rightarrow & \mathbf{E} + E + E * E \\ \Rightarrow & x + \mathbf{E} + E * E \end{aligned}$$

# Example: derivation

## Example (well-formed expressions over $x, +, *, (, )$ )

Let  $G_E = (N, \Sigma, P, E)$  with

- ▶  $N = \{E\}$
- ▶  $\Sigma = \{(, ), +, *, x\}$
- ▶  $P$ :

- 1  $E \rightarrow x$
- 2  $E \rightarrow (E)$
- 3  $E \rightarrow E + E$
- 4  $E \rightarrow E * E$

The following is a derivation of  $x + x + x * x$  (with the replaced symbol printed bold):

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## Example (well-formed expressions over $x, +, *, (, )$ )

Let  $G_E = (N, \Sigma, P, E)$  with

- ▶  $N = \{E\}$
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# Rightmost and leftmost Derivations

## Definition (rightmost/leftmost)

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## Example

- ▶ The derivation on the previous slide is neither leftmost nor rightmost.
- ▶ A **rightmost derivation** is:

$$\mathbf{E} \Rightarrow E + \mathbf{E} \Rightarrow E + E + \mathbf{E} \Rightarrow E + E + E * \mathbf{E} \Rightarrow E + E + \mathbf{E} * x \Rightarrow E + \mathbf{E} + x * x \Rightarrow \mathbf{E} + x + x * x \Rightarrow x + x + x * x$$

## Definition (parse tree)

A **parse tree** for a derivation in a grammar  $G = (N, \Sigma, P, S)$  is an ordered, labelled tree with the following properties:

- ▶ Each node is labelled with a symbol from  $N \cup \Sigma$
- ▶ The root of the tree is labelled with the start symbol  $S$ .
- ▶ Each inner node is labelled with a single non-terminal symbol from  $V_N$
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- 
- ▶ The parse tree represents a derivation of the word formed by the labels of the leaf nodes
  - ▶ It abstracts from the order in which productions are applied.

# Example: parse trees

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The derivation  $\mathbf{E} \Rightarrow E + \mathbf{E} \Rightarrow E + E + \mathbf{E} \Rightarrow E + E + E * \mathbf{E} \Rightarrow E + E + \mathbf{E} * x \Rightarrow E + \mathbf{E} + x * x \Rightarrow \mathbf{E} + x + x * x \Rightarrow x + x + x * x$  can be represented by a sequence of parse trees:

# Example: parse trees

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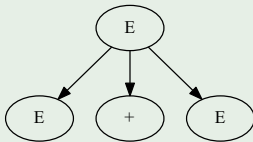
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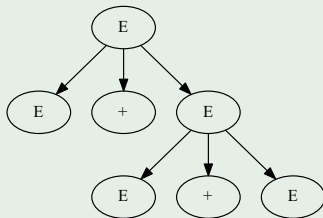




# Example: parse trees

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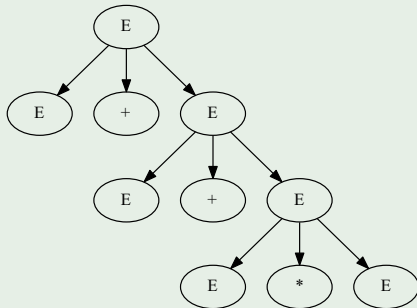
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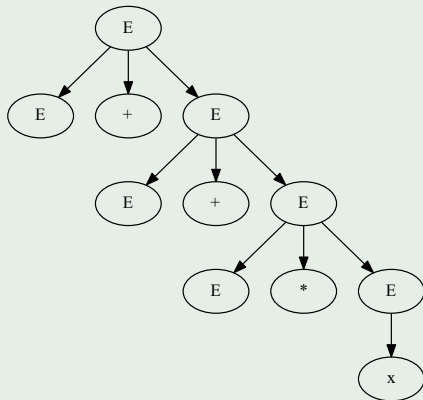
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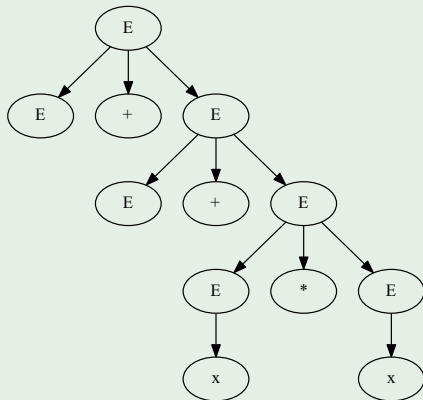
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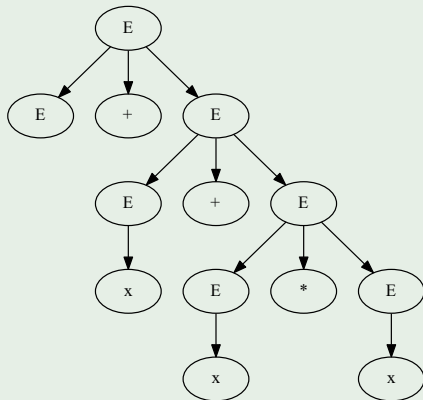
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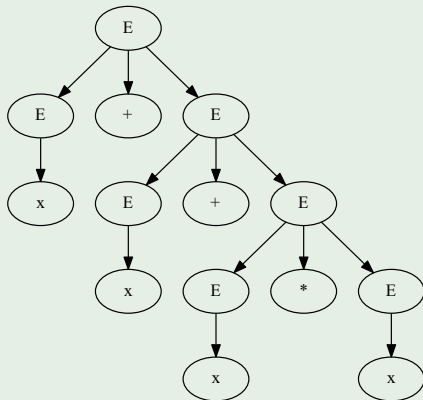
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A grammar  $G = (N, \Sigma, P, S)$  is **ambiguous** if it has multiple different parse trees for a word  $w$  in  $L(G)$ .

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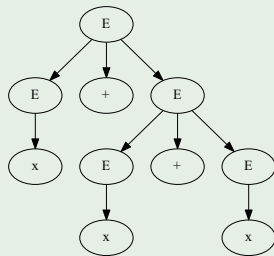
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$G_E$  is ambiguous since it has several parse trees for  $x + x + x$ .

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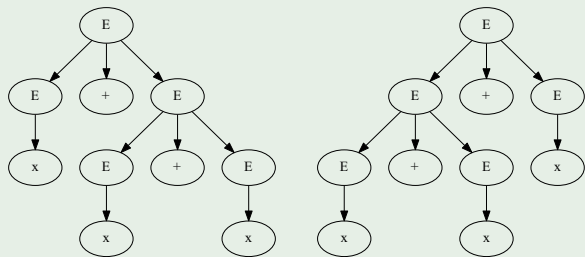
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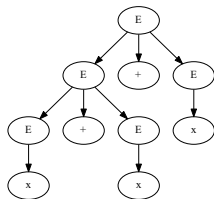
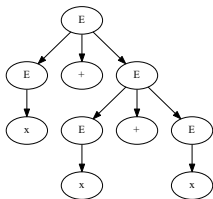
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## Exercise: Ambiguity is worse...

Consider the grammar  $G_E$  and its parse trees for  $x + x + x$ .

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- ▶ Provide a rightmost derivation for the right tree. A-M A-M
- ▶ Provide a rightmost derivation for the left tree. A-M N-Z
- ▶ Provide a leftmost derivation for the left tree. N-Z A-M
- ▶ Provide a leftmost derivation for the right tree. N-Z N-Z

# Exercise: Eliminating Ambiguity

Consider  $G_E$  with the following productions:

1  $E \rightarrow x$

2  $E \rightarrow (E)$

3  $E \rightarrow E + E$

4  $E \rightarrow E * E$

Define a grammar  $G'$  with  $L(G) = L(G')$  that is not ambiguous, that respects that  $*$  has a higher precedence than  $+$ , and that respects left-associativity for all operators.

# Abstract Syntax Trees

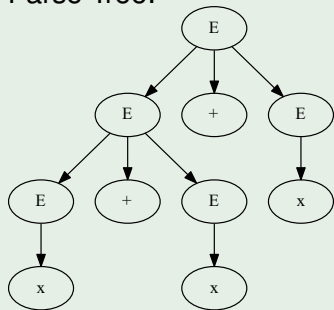
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Parse Tree:

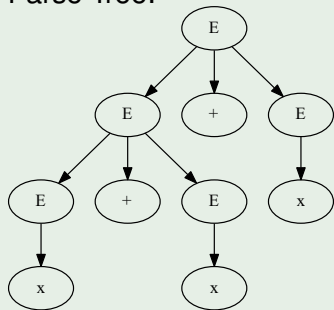


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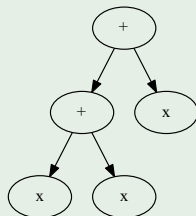
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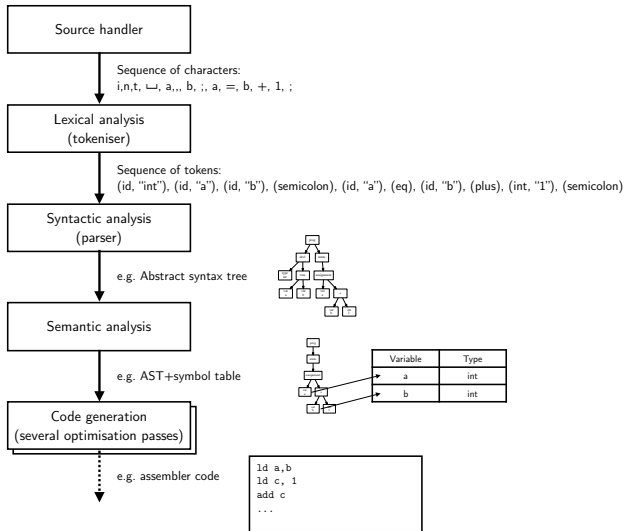
Parse Tree:



Corresponding AST:



# High-Level Architecture of a Compiler



# Syntactic Analysis/Parsing

- ▶ Description of the language with a **context-free grammar**
- ▶ Parsing:
  - ▶ Try to build a **parse tree**/abstract syntax tree (AST)
  - ▶ Parse tree unambiguously describes structure of a program
  - ▶ AST reflects abstract syntax (can e.g. drop parenthesis)
- ▶ Methods:
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**Result:** Abstract Syntax Tree

# Semantic Analysis

- ▶ Analyze static properties of the program
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**Result:** Annotated AST

# Optimization

- ▶ Transform Abstract Syntax Tree to generate better code
  - ▶ Smaller
  - ▶ Faster
  - ▶ Both
- ▶ Mechanisms
  - ▶ Common sub-expression elimination
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- ▶ Convert optimized AST into low-level code
- ▶ Target languages:
  - ▶ Assembly code
  - ▶ Machine code
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**Result:** Program in target language

# Summary: parsers

## Parsers

- ▶ recognise structure of programs
- ▶ receive **tokens** from scanner
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End lecture 16

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Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

Turing Machines

Unrestricted Grammars

Linear Bounded Automata

Properties of Type-0-languages

Lecture-specific material

Bonus Exercises

Selected Solutions

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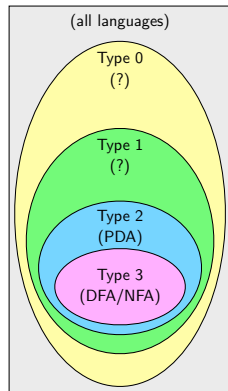
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## Turing Machines

# Turing Machine: Motivation

Four classes of languages described by grammars and equivalent machine models:

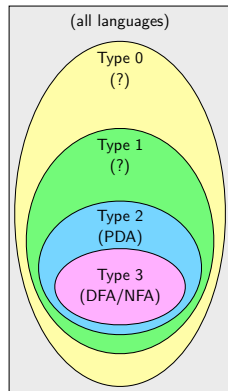
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We need a machine model that is more powerful than PDAs:

**Turing Machines**

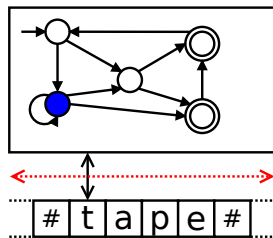
# Turing Machine: history

- ▶ proposed in 1936 by Alan Turing
  - ▶ paper: *On computable numbers, with an application to the Entscheidungsproblem*
  - ▶ uses the TM to show that satisfiability of first-order formulas is **undecidable**
- ▶ model of a **universal computer**
  - ▶ very simple (and thus easy to describe formally)
  - ▶ but as powerful as any conceivable machine





# Turing Machine: conceptual model



- ▶ medium: unlimited **tape** (bidirectional)
  - ▶ initially contains input (and blanks #)
  - ▶ TM can read and **write** tape
  - ▶ TM can **move arbitrarily** over tape
  - ▶ serves for input, working, output
  - ▶ **output** possible
- ▶ transition relation
  - ▶ read and write current position
  - ▶ moving instructions (l, r, n)
- ▶ acceptance condition
  - ▶ **final state** is reached
  - ▶ no transitions possible
- ▶ commonalities with FA
  - ▶ control unit (finite set of states),
  - ▶ initial and final states
  - ▶ input alphabet

# Transitions in Turing Machines

$$\Delta \subseteq Q \times \Gamma \times \Gamma \times \{l, n, r\} \times Q$$

- ▶ TM is in state
- ▶ reads tape symbol from current position
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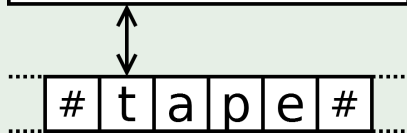
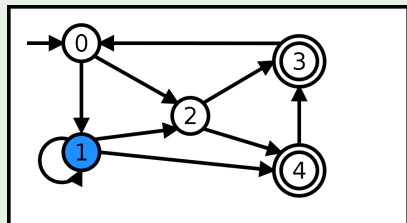
$p a \rightarrow a r p$   
 $p b \rightarrow b r p$   
 $p \# \rightarrow \# r p$

A transition  $p, a, b, l, q$  can also be written as

$$p \ a \ \rightarrow \ b \ l \ q$$

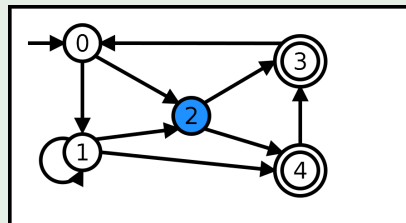
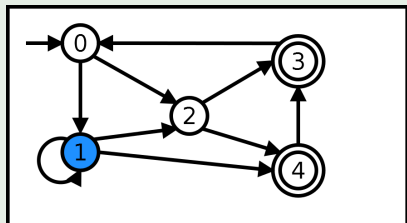
# Example: transition

Example (transition  $1, t \rightarrow c, r, 2$ )



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## Definition (Turing Machine)

A **Turing Machine** (TM) is a 6-tuple  $(Q, \Sigma, \Gamma, \Delta, q_0, F)$  where

- ▶  $Q, \Sigma, q_0, F$  are defined as for NFAs,
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# Turing Machine: formal definition

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Note:

- ▶  $\Gamma$  (the tape alphabet) can contain additional characters
- ▶ This does not increase the power, but makes finding TMs easier



## Definition (configuration)

A **configuration**  $c = \alpha q \beta$  of a Turing Machine is given by

- ▶ the current state  $q$
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A configuration  $c = \alpha q \beta$  is **accepting** if  $q \in F$ .

# Configurations of TMs

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- ▶ the tape content  $\beta$  starting with the position of the head (except unlimited # sequences)

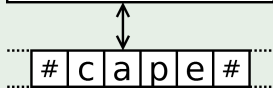
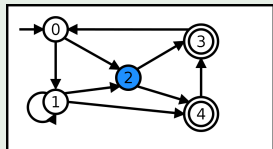
A configuration  $c = \alpha q \beta$  is **accepting** if  $q \in F$ .

A configuration  $c$  is a **stop configuration** if there are no transitions from  $c$ .

# Example: configuration

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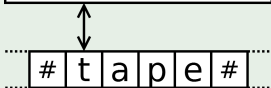
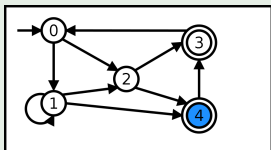
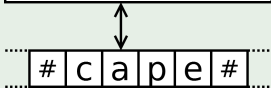
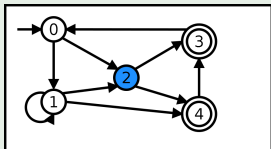
## Example (configurations)



- ▶ This TM is in the configuration  $c2ape$ .

# Example: configuration

## Example (configurations)



- ▶ This TM is in the configuration  $c2ape$ .
- ▶ The configuration  $4tape$  is accepting.
- ▶ If there are no transitions  $4, t \rightarrow \dots$ ,  $4tape$  also is a stop configuration.

## Definition (computation, acceptance)

A **computation** of a TM  $\mathcal{M}$  on a word  $w$  is a sequence of configurations (according to the transition function) of configurations of  $\mathcal{M}$ , starting from  $q_0w$ .

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$\mathcal{M}$  **accepts**  $w$  if there exists a computation of  $\mathcal{M}$  on  $w$  that leads to an accepting stop configuration.



## Exercise: Turing Machines

Let  $\Sigma = \{a, b\}$  and  $L = \{w \in \Sigma^* \mid |w|_a \text{ is even}\}$ .

- ▶ Give a TM  $\mathcal{M}$  that accepts (exactly) the words in  $L$ .
- ▶ Give the computation of  $\mathcal{M}$  on the words *abbab* and *bbab*.

# Example: TM for $a^n b^n c^n$

$\mathcal{M} = (Q, \Sigma, \Gamma, \Delta, \text{start}, \{f\})$  with

- ▶  $Q = \{\text{start, findb, findc, check, back, end, f}\}$
- ▶  $\Sigma = \{a, b, c\}$  and  $\Gamma = \Sigma \cup \{\#, x, y, z\}$

state	read	write	move	state	state	read	write	move	state
start	#	#	n	f	back	z	z	l	back
start	a	x	r	findb	back	b	b	l	back
findb	a	a	r	findb	back	y	y	l	back
findb	y	y	r	findb	back	a	a	l	back
findb	b	y	r	findc	back	x	x	r	start
findc	b	b	r	findc	end	z	z	l	end
findc	z	z	r	findc	end	y	y	l	end
findc	c	z	r	check	end	x	x	l	end
check	c	c	l	back	end	#	#	n	f
check	#	#	l	end					

## Exercise: Turing Machines (2)

- a) Simulate the computations of  $\mathcal{M}$  on  $aabbcc$  and  $aabc$ .
- b) Develop a Turing Machine  $\mathcal{P}$  accepting  $L_{\mathcal{P}} = \{w cw \mid w \in \{a, b\}^*\}$ .
- c) How do you have to modify  $\mathcal{P}$  if you want to recognise inputs of the form  $ww$ ?

# Turing Machines with several tapes

- ▶ A  $k$ -tape TM has  $k$  tapes on which the heads can move independently.
- ▶  $\Delta \subseteq Q \times \Gamma^k \times \Gamma^k \times \{r, l, n\}^k \times Q$
- ▶ It is possible to simulate a  $k$ -tape TM with a (1-tape) TM:
  - ▶ use alphabet  $\Gamma^k \times \{X, \#\}^k$
  - ▶ the first  $k$  language elements encode the tape content, the remaining ones the positions of the heads.

## Reminder

- ▶ just like FAs and PDAs, TMs can be deterministic or non-deterministic, depending on the transition relation.
- ▶ for non-deterministic TMs, the machine accepts  $w$  if there **exists** a sequence of transitions leading to an accepting stop configuration.

## Simulating non-deterministic TMs

Theorem (equivalence of deterministic and non-deterministic TMs)

*Deterministic TMs can simulate computations of non-deterministic TMs; i.e. they describe the same class of languages.*

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## Proof.

Use a 3-tape TM:

- ▶ tape 1 stores the input  $w$
- ▶ tape 2 enumerates all possible sequences of non-deterministic choices (for all non-deterministic transitions)
- ▶ tape 3 encodes the computation on  $w$  with choices stored on tape 2.



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## Theorem (equivalence of TMs and unrestricted grammars)

*The class of languages that can be accepted by a Turing Machine is exactly the class of languages that can be generated by unrestricted Chomsky grammars.*

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## Proof.

- 1 simulate grammar derivations with a TM
- 2 simulate a TM computation with a grammar



# Simulating a Type-0-grammar $G$ with a TM

Use a non-deterministic 2-tape TM:

- ▶ tape 1 stores input word  $w$
- ▶ tape 2 simulates the derivations of  $G$ , starting with  $S$ 
  - ▶ (non-deterministically) choose a position
  - ▶ if the word starting at the position, matches  $\alpha$  of a rule  $\alpha \rightarrow \beta$ , apply the rule
    - ▶ move tape content if necessary
    - ▶ replace  $\alpha$  with  $\beta$
  - ▶ compare content of tape 2 with tape 1
    - ▶ if they are equal, accept
    - ▶ otherwise continue

# Simulating a TM with a Type-0-grammar

Goal: transform TM  $\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, q_0, F)$  into grammar  $G$

Technical difficulty:

- ▶  $\mathcal{A}$  receives word as input **at the start**, possibly modifies it, then possibly accepts.
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  - ▶  $G$  starts with  $S$ , applies rules, possibly generating  $w$  **at the end**.
- 1** generate initial configuration  $q_0w \in \Sigma^*$  with blanks left and right
  - 2** simulate the computation of  $\mathcal{A}$  on  $w$

$$(p, a, b, r, q) \rightsquigarrow pa \rightarrow bq$$

$$(p, a, b, l, q) \rightsquigarrow cpa \rightarrow qcb \text{ (for all } c \in \Gamma)$$

$$(p, a, b, n, q) \rightsquigarrow pa \rightarrow qb$$

- 3** if an accepting stop configuration is reached, recreate  $w$ 
  - ▶ requires a more complex alphabet with a “backup track”

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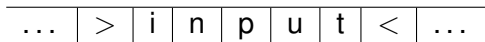
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# Linear bounded automata

- ▶ context-sensitive grammars do not allow for contracting rules
- ▶ a **linear bounded automaton (LBA)** is a TM that only uses the space originally occupied by the input  $w$ .
- ▶ limits of  $w$  are indicated by markers that cannot be passed by the read/write head

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# Equivalence of cs. grammars and LBAs

Transformation of cs. grammar  $G$  into LBA:

- ▶ as for Type-0-grammar: use 2-tape-TM
  - ▶ input on tape 1
  - ▶ simulate operations of  $G$  on tape 2
- ▶ since the productions of  $G$  are non-contracting, words longer than  $w$  need not be considered

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Transformation of LBA  $\mathcal{A}$  into cs. grammar:

- ▶ similar to construction for TM:
  - ▶ generate  $w$  **without blanks**
  - ▶ simulate operation of  $\mathcal{A}$  on  $w$ 
    - ▶ rules are non-contracting ✓

# Closure properties: regular operations

## Theorem (closure under $\cup, \cdot, *$ )

*The class of languages described by context-sensitive grammars is closed under  $\cup, \cdot, *$ .*

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*The class of languages described by context-sensitive grammars is closed under  $\cup, \cdot, *$ .*

## Proof.

Concatenation and Kleene-star are more complex than for cf. grammars because the context can influence rule applicability.

- ▶ rename NTSs (as for cf. grammars)
- ▶ only allow NTSs as context
- ▶ only allow productions of the kind
  - ▶  $N_1N_2 \dots N_k \rightarrow M_1M_2 \dots M_j$
  - ▶  $N \rightarrow a$



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## Proof.

- ▶ use a 2-tape-LBA
- ▶ simulate computation of  $\mathcal{A}_1$  on tape 1,  $\mathcal{A}_2$  on tape 2
- ▶ accept if both  $\mathcal{A}_1$  and  $\mathcal{A}_2$  accept □

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- ▶ shown in 1988



# Context-sensitive grammars: decision problems

Theorem (Word problem for cs. languages)

*The **word** problem for cs. languages is **decidable**.*

## Theorem (Word problem for cs. languages)

The *word* problem for cs. languages is *decidable*.

## Proof.

- ▶  $N$ ,  $\Sigma$  and  $P$  are finite
- ▶ rules are non-contracting
- ▶ for a word of length  $n$  only a finite number of derivations up to length  $n$  has to be considered.



## Context-sensitive grammars: decision problems (cont')

### Theorem (Emptiness problem for cs. languages)

*The **emptiness** problem for cs. languages is **undecidable**.*

### Proof.

Also follows from undecidability of Post's correspondence problem. □

## Context-sensitive grammars: decision problems (cont')

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The *emptiness* problem for cs. languages is *undecidable*.

### Proof.

Also follows from undecidability of Post's correspondence problem. □

### Theorem (Equivalence problem for cs. languages)

The *equivalence* problem for cs. languages is *undecidable*.

### Proof.

If this problem was decidable for cs. languages, it would also be decidable for cf. languages (since every cf. language is also cs.). □

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# The universal Turing Machine $\mathcal{U}$

- ▶  $\mathcal{U}$  is a TM that simulates other Turing Machines
- ▶ since TMs have finite alphabets and state sets, they can be encoded by a (binary) alphabet by an encoding function  $c()$
- ▶ Input:
  - ▶ encoding  $c(\mathcal{A})$  of a TM  $\mathcal{A}$  on tape 1
  - ▶ encoding  $c(w)$  of an input word  $w$  for  $\mathcal{A}$  on tape 2
- ▶ with input  $c(\mathcal{A})$  and  $c(w)$ ,  $\mathcal{U}$  behaves exactly like  $\mathcal{A}$  on  $w$ :
  - ▶  $\mathcal{U}$  accepts iff  $\mathcal{A}$  accepts
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**Every solvable problem can be solved in software.**

# Operation of $\mathcal{U}$

- 1 encode initial configuration
  - ▶ tape on lhs of head
  - ▶ state
  - ▶ tape on rhs of head
- 2 use  $c(\mathcal{A})$  to find a transition from the current configuration
- 3 modify the current configuration accordingly
- 4 accept if  $\mathcal{A}$  accepts
- 5 stop if  $\mathcal{A}$  stops
- 6 otherwise, continue with step 2



# The Halting problem

## Definition (halting problem)

For a TM  $\mathcal{A} = (Q, \Sigma, \Gamma, q_0, \Delta, F)$  and a word  $w \in \Sigma^*$ , does  $\mathcal{A}$  halt (i.e. reach a stop configuration) with input  $w$ ?

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decision procedure for HP: let  $\mathcal{H}1$  and  $\mathcal{H}2$  run in parallel

- 1  $\mathcal{U}$  (almost) does what  $\mathcal{H}1$  needs to do.
- 2 Difficult:  $\mathcal{H}2$  needs to detect that that  $\mathcal{A}$  does not terminate.
  - ▶ infinite tape  $\leadsto$  infinite number possible configurations
  - ▶ recognising repeated configurations not sufficient.

# Undecidability of the halting problem

Assumption: there is a TM  $\mathcal{H}2$  which, given  $c(\mathcal{A})$  and  $c(w)$  as input

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- 3 afterwards  $\mathcal{S}$  operates like  $\mathcal{H}_2$

## Computation of $\mathcal{S}$ with input $c(\mathcal{S})$

Reminder:  $\mathcal{S}$  accepts  $c(\mathcal{A})$  iff  $\mathcal{A}$  does **not** accept  $c(\mathcal{A})$ .

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This implies:

**1** There is no such TM  $\mathcal{S}$ .



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Reminder:  $\mathcal{S}$  accepts  $c(\mathcal{A})$  iff  $\mathcal{A}$  does **not** accept  $c(\mathcal{A})$ .

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- Ca**  $\mathcal{S}$  rejects  $c(\mathcal{S})$ . Since  $\mathcal{S}$  accepts exactly the encodings of those TMs that reject their own encoding, this implies that  $\mathcal{S}$  accepts the input  $c(\mathcal{S})$ . ⚡

This implies:

- 1** There is no such TM  $\mathcal{S}$ .
- 2** There is no TM  $\mathcal{H}2$ .

# Computation of $\mathcal{S}$ with input $c(\mathcal{S})$

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## Theorem (Turing 1936)

*The halting problem is undecidable.*

## Theorem (Decision problems for Turing Machines)

*The word problem, the emptiness problem, and the equivalence problem are undecidable.*

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*The word problem, the emptiness problem, and the equivalence problem are undecidable.*

## Proof.

If any of these problems were decidable, one could easily derive a decision procedure for the halting problem. □

# Closure properties

## Theorem (closure under $\bar{\phantom{x}}$ )

*The class of languages accepted by Turing Machines is **not closed** under complement.*

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*The class of languages accepted by TMs is **closed** under  $\cup, \cdot, *, \cap$ .*

## Proof.

Analogous to Type-1-grammars / LBAs.



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Challenge of the proof:  
show for all possible (infinitely many) TMs that none of them can decide the halting problem.

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show for all possible (infinitely many) TMs that none of them can decide the halting problem.

TM	input	$c(A)$	$c(B)$	$c(C)$	$c(D)$	$c(E)$	$\dots$
$A$		$\times$					
$B$			$\times$				
$C$				$\times$			
$D$					$\times$		
$E$						$\times$	
$\dots$							$\dots$

## Further diagonalisation arguments

Theorem (Cantor diagonalisation, 1891)

*The set of real numbers is uncountable.*

Theorem (Epimenides paradox, 6th century BC)

*Epimenides [the Cretan] says: “[All] Cretans are always liars.”*

Theorem (Russell's paradox, 1903)

$R := \{T \mid T \notin T\}$  Does  $R \in R$  hold?

Theorem (Gödel's incompleteness theorem, 1931)

*Construction of a sentence in 2nd order predicate logic which states that itself cannot be proved.*

## Is this important?

- ▶ What is so bad about not being able to decide if a TM halts?
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**Does it matter in practice?**

## It does not only affect halting

Halting is a fundamental property.

If halting cannot be decided, what can be?

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**non-trivial** satisfied by some TMs, not satisfied by others

**semantic** referring to the accepted language

# Undecidability of semantic properties

Example (Property  $E$ : TM accepts the set of prime numbers  $P$ )

If  $E$  is decidable, then so is the halting problem for  $\mathcal{A}$  and an input  $w_{\mathcal{A}}$ .  
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- 2 decide if  $w_{\mathcal{E}} \in P$

Check if  $\mathcal{E}$  accepts the set of prime numbers:

yes  $\leadsto \mathcal{A}$  halts with input  $w_{\mathcal{A}}$       no  $\leadsto \mathcal{A}$  does not halt on input  $w_{\mathcal{A}}$

# It does not only affect Turing Machines

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**No interesting property is decidable  
for any powerful programming language!**

# Undecidable problems in practice

- software development** Does the program match the specification?
- debugging** Does the program have a memory leak?
- malware** Does the program harm the system?
- education** Does the student's TM compute the same function as the teacher's TM?
- formal languages** Do two cf. grammars generate the same language?
- mathematics** Hilbert's tenth problem: find integer solutions for a polynomial with several variables
- logic** Satisfiability of formulas in first-order predicate logic

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***Yes, it does matter!***

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there will always be cases in which an incorrect answer or none at all is given.

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- ▶ interactive programs

# Turing Machines: summary

- ▶ Halting problem: does TM  $\mathcal{A}$  halt on input  $w$ ?
- ▶ Turing: no TM can decide the halting problem.
- ▶ Rice: no TM can decide any non-trivial semantic property of TMs.
- ▶ Church-Turing: this holds for every powerful machine model.
- ▶ No interesting problem of programs in any powerful programming language is decidable.

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- ▶ Church-Turing: this holds for every powerful machine model.
- ▶ No interesting problem of programs in any powerful programming language is decidable.

Consequences:

- ☹ Computers cannot take all work away from computer scientists.
- 😊 Computers will never make computer scientists redundant.

# Property overview

property	regular (Type 3)	context-free (Type 2)	context-sens. (Type 1)	unrestricted (Type 0)
closure				
$\cup, \cdot, *$	✓	✓	✓	✓
$\cap$	✓	✗	✓	✓
$\_$	✓	✗	✓	✗
decidability				
word	✓	✓	✓	✗
emptiness	✓	✓	✗	✗
equiv.	✓	✗	✗	✗
deterministic equivalent to non-det.	✓	✗	?	✓

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End lecture 17



**This is the End...**

# Outline

Introduction

Regular Languages and Finite Automata

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

## Lecture-specific material

Lecture 1

Lecture 2

Lecture 3

Lecture 4

Lecture 5

Lecture 6

Lecture 7

Lecture 8

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Lecture 10

Lecture 11

Lecture 12

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Selected Solutions

## **Lecture-specific material**

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# Goals for Lecture 1

- ▶ Getting acquainted
- ▶ Clarifying practical issues
- ▶ Course outline and motivation
  - ▶ Formal languages
  - ▶ Language classes
  - ▶ Grammars
  - ▶ Automata
  - ▶ Questions
  - ▶ Applications

# Practical Issues

- ▶ Lecture times (usually, check RAPLA)
  - ▶ Wednesday 13:30
  - ▶ Thursday 16:00
  - ▶ Friday 12:15
  - ▶ Really, check RAPLA - we only have 18 lectures, not 33.
- ▶ Final exam
- ▶ Written exam (Open-Book or nearly so)
- ▶ Probably week 48 (November 23rd to 27th)
- ▶ Possibly online, possibly in the Audimax

# Summary Lecture 1

- ▶ Clarifying practical issues
  - ▶ You need running `flex`, `bison`, C compiler, editor!
- ▶ Course outline and motivation
  - ▶ Formal languages
  - ▶ Language classes
  - ▶ Grammars
  - ▶ Automata
  - ▶ Questions
  - ▶ Applications

# Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?



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Lecture 12

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Lecture 14

Lecture 15

Lecture 16

Lecture 17

Lecture 18

Bonus Exercises

Selected Solutions

## Goals for Lecture 2

- ▶ Review of last lecture
- ▶ Formal basics of formal languages
- ▶ Operations on words
- ▶ Operations on languages
- ▶ Introduction to regular expressions
  - ▶ Examples
  - ▶ Formal definition

- ▶ Introduction
  - ▶ Language classes
  - ▶ Grammars
  - ▶ Automata
  - ▶ Applications
- ▶ `flex` and `bison` are available via most Open Source install methods (if not installed by default)
  - ▶ E.g. `apt-get install bison` (Debian/Ubuntu)
  - ▶ E.g. `sudo port install flex` (MacPorts)
- ▶ Check out <http://dinosaur.compilertools.net/>
  - ▶ Documentation, sources
- ▶ `flex` **direct**: <https://github.com/westes/flex>
- ▶ `bison` **direct**: <https://www.gnu.org/software/bison/>

# Summary

- ▶ Formal languages
  - ▶ Alphabets
  - ▶ Words
  - ▶ Languages
  - ▶ Examples of languages
- ▶ Operations on words and languages
  - ▶ Concatenation
  - ▶ Power
  - ▶ Kleene star
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Remark: Formal languages are *sets*, and hence we can also apply set operations like  $\cup$  (union),  $\cap$  (intersection),  $\bar{\phantom{x}}$  (complement) to them!

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Bonus Exercises

Selected Solutions

# Goals for Lecture 3

- ▶ Review of last lecture
- ▶ Understanding regular expressions
- ▶ Regular expression algebra
  - ▶ Equivalences on regular expressions
  - ▶ Simplifying REs



# Review (1)

- ▶ Formal languages
  - ▶ Finite **alphabet**  $\Sigma$  of symbols/letters
  - ▶ **Words** are finite sequences of letters from  $\Sigma$
  - ▶ **Languages** are (finite or infinite) sets of words
- ▶ Words - properties and operations
  - ▶  $|w|, |w|_a, w[k]$
  - ▶  $w_1 \cdot w_2, w^n$

# Review (2)

- ▶ Interesting languages
  - ▶ Binary representations of natural numbers
  - ▶ Binary representations of prime numbers
  - ▶ C functions (over strings)
  - ▶ C functions with input/output pairs
- ▶ Operations on Languages
  - ▶ Product  $L_1 \cdot L_2$ : Concatenation of one word from each language
  - ▶ Power  $L^n$ : Concatenation of  $n$  words from  $L$
  - ▶ Kleene Star:  $L^*$ : Concat any number of words from  $L$

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- ▶ Regular expressions  $R_\Sigma$ 
  - ▶ Base cases:
    - ▶  $L(\emptyset) = \{\}$
    - ▶  $L(\epsilon) = \{\epsilon\}$
    - ▶  $L(a) = \{a\}$  for each  $a \in \Sigma$
  - ▶ Complex cases:
    - ▶  $L(r_1 + r_2) = L(r_1) \cup L(r_2)$
    - ▶  $L(r_1 \cdot r_2) = L(r_1 r_2) = L(r_1) \cdot L(r_2)$
    - ▶  $L(r^*) = L(r)^*$
    - ▶  $L((r)) = L(r)$  (brackets are used to group expressions)

- ▶ Regular expression algebra
  - ▶ REs are equivalent if they describe the same language
  - ▶ REs can be simplified applying equivalences
  - ▶ Recursive definitions: Lemmas of Arden/Salomaa

- ▶ What was the best part of today's lecture?
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Bonus Exercises

Selected Solutions

# Goals for Lecture 4

- ▶ Review of last lecture
- ▶ Warmup exercise
- ▶ Finite Automata
  - ▶ Graphical representation
  - ▶ Formal definition
  - ▶ Language recognized by an automata
  - ▶ Tabular representation
  - ▶ Exercises



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    - ▶  $L(r_1 \cdot r_2) = L(r_1 r_2) = L(r_1) \cdot L(r_2)$
    - ▶  $L(r^*) = L(r)^*$
    - ▶  $L((r)) = L(r)$  (brackets are used to group expressions)

## Review (2)

- ▶ Regular expression algebra
  - ▶ REs are equivalent if they describe the same language
  - ▶ REs can be simplified applying equivalences
  - ▶ 17 equivalences
    - ▶ Commutativity of  $+$ , associativity of  $+$ ,  $\cdot$ , distributivity (!)
    - ▶ Arden/Salomaa
    - ▶ ...

# Warmup Exercise

- ▶ Assume  $\Sigma = \{a, b\}$ 
  - ▶ Find a regular expression for the language  $L_1$  of all words over  $\Sigma$  with at least 3 characters and where the third character is a  $a$ .
  - ▶ Describe  $L_1$  formally (i.e. as a set)
  - ▶ Find a regular expression for the language  $L_2$  of all words over  $\Sigma$  with at least 3 characters and where the third character is the same as the third-last character
  - ▶ Describe  $L_2$  formally.

# Warmup Exercise

- ▶ Assume  $\Sigma = \{a, b\}$ 
  - ▶ Find a regular expression for the language  $L_1$  of all words over  $\Sigma$  with at least 3 characters and where the third character is a  $a$ .
  - ▶ Describe  $L_1$  formally (i.e. as a set)
  - ▶ Find a regular expression for the language  $L_2$  of all words over  $\Sigma$  with at least 3 characters and where the third character is the same as the third-last character
  - ▶ Describe  $L_2$  formally.

- ▶ REs revisited (with exercise)
- ▶ Finite Automata
  - ▶ Graphical representation
  - ▶ Formal definition
  - ▶ Language recognized by an automata
  - ▶ Tabular representation
  - ▶ Exercises

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

# Goals for Lecture 5

- ▶ Review of last lecture
- ▶ Introduction to Nondeterministic Finite Automata
  - ▶ Definitions
  - ▶ Exercises
- ▶ Equivalency of deterministic and nondeterministic finite automata
  - ▶ Converting NFAs to DFAs
  - ▶ Exercises



## ▶ Finite Automata

- ▶ Graphical representation
- ▶ Formal definition
  - ▶  $Q$ : Set of states
  - ▶  $\Sigma$ : Alphabet
  - ▶  $\delta$ : Transition function
  - ▶  $q_0$ : Initial state
  - ▶  $F$ : Final (accepting) states
- ▶ Language recognized by an automata
  - ▶  $w \in L(A)$  if  $\delta'(w) \in F$  or there exist an accepting run of  $A$  for  $w$
- ▶ Tabular representation
- ▶ Exercises

## ▶ Finite Automata

- ▶ Graphical representation
- ▶ Formal definition
  - ▶  $Q$ : Set of states
  - ▶  $\Sigma$ : Alphabet
  - ▶  $\delta$ : Transition function
  - ▶  $q_0$ : Initial state
  - ▶  $F$ : Final (accepting) states
- ▶ Language recognized by an automata
  - ▶  $w \in L(A)$  if  $\delta'(w) \in F$  or there exist an accepting run of  $A$  for  $w$
- ▶ Tabular representation
- ▶ Exercises

- ▶ Nondeterministic Finite Automata
  - ▶  $Q$ : Set of states
  - ▶  $\Sigma$ : Alphabet
  - ▶  $\Delta$ : Transition *relation* (with  $\epsilon$ -transitions!)
  - ▶  $q_0$ : Initial state
  - ▶  $F$ : Final (accepting) states
- ▶ NFAs and DFAs accept the same class of languages!
- ▶ Converting NFA to DFA
  - ▶ States of  $\text{det}(\mathcal{A})$  are elements of  $2^Q$
  - ▶ ...

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

# Goals for Lecture 6

- ▶ Review of last lecture
- ▶ Equivalency of regular expressions and NFAs
  - ▶ Construction of an NFA from a regular expression
  - ▶ Extraction of an RE from a DFA

- ▶ Nondeterministic Finite Automata
  - ▶  $Q$ : Set of states
  - ▶  $\Sigma$ : Alphabet
  - ▶  $\Delta$ : Transition *relation* (with  $\varepsilon$ -transitions!)
  - ▶  $q_0$ : Initial state
  - ▶  $F$ : Final (accepting) states
- ▶ NFAs and DFAs accept the same class of languages!
- ▶ Converting NFA to DFA
  - ▶ States of  $\det(\mathcal{A})$  are elements of  $2^Q$
  - ▶ ...

- ▶ Equivalency of regular expressions and NFAs
  - ▶ Construction of an NFA from a regular expression
    - ▶ Base cases each result in a trivial 2-state automaton
    - ▶ Compose automata for composite regular expressions
    - ▶ *Glue* automata together with  $\epsilon$ -Transitions
  - ▶ Extraction of an RE for a DFA
    - ▶ Determine system of equations
    - ▶ For each state add one alternative for each transition
    - ▶ For accepting states add  $\epsilon$ .
    - ▶ Solve the system of equations, handling loops with Arden's lemma



- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

# Goals for Lecture 7

- ▶ Review of last lecture
- ▶ Minimizing DFAs
  - ▶ ... and a first application
- ▶ Exercises

# Review

- ▶ Central result: REs, NFAs and DFAs describe the same class of languages (namely **regular languages**)
  - ▶ Proof via constructive algorithms
    - ▶ NFA to DFA already done
    - ▶ RE to NFA and DFA to RE new
- ▶ Construction of an NFA from a regular expression
  - ▶ Base cases each result in a trivial 2-state automaton
  - ▶ Compose automata for composite regular expressions
  - ▶ *Glue* automata together with  $\epsilon$ -Transitions
- ▶ Extraction of an RE for a DFA
  - ▶ Determine system of equations describing language accepted at each state
    - ▶ For each state add one alternative for each transition
    - ▶ For accepting states add  $\epsilon$ .
  - ▶ Solve the system of equations, handling loops with Arden's lemma

# Homework assignment

- ▶ Get access to an operational Linux/UNIX environment
  - ▶ You can install VirtualBox (<https://www.virtualbox.org>) and then install e.g. Ubuntu (<http://www.ubuntu.com/>) on a virtual machine
  - ▶ For Windows, you can install the **complete** UNIX emulation package Cygwin from <http://cygwin.com>
  - ▶ For MacOS, you can install `fink` (<http://fink.sourceforge.net/>) or `MacPorts` (<https://www.macports.org/>) and the necessary tools
- ▶ You will need at least `flex`, `bison`, `gcc`, `grep`, `sed`, `AWK`, `make`, and a good text editor of your choice

- ▶ Minimizing DFAs
  - ▶ Identify and merge equivalent states
  - ▶ Result is unique (up to names of states)
  - ▶ Equivalency of REs can be decided by comparison of corresponding minimal DFAs
- ▶ Homework: Get ready for `flexing...`

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions



# Goals for Lecture 8

- ▶ Review of last lecture
- ▶ Beyond regular languages: The Pumping Lemma
  - ▶ Motivation/Lemma
  - ▶ Application of the lemma
  - ▶ Implications
- ▶ Properties of regular languages
  - ▶ Which operations leave regular languages regular?

- ▶ Minimizing DFAs
  - ▶ Identify and merge equivalent states
  - ▶ Result is unique (up to names of states)
  - ▶ Equivalency of REs can be decided by comparison of corresponding minimal DFAs

## Reminder: Homework assignment

- ▶ Install an operational UNIX/Linux environment on your computer
  - ▶ You can install VirtualBox (<https://www.virtualbox.org>) and then install e.g. Ubuntu (<http://www.ubuntu.com/>) on a virtual machine
  - ▶ For Windows, you can install the **complete** UNIX emulation package Cygwin from <http://cygwin.com>
  - ▶ For MacOS, you can install `fink` (<http://fink.sourceforge.net/>) or MacPorts (<https://www.macports.org/>) and the necessary tools
- ▶ You will need at least `flex`, `bison`, `gcc`, `grep`, `sed`, `AWK`, `make`, and a good text editor of your choice

- ▶ Beyond regular languages: The Pumping Lemma
  - ▶ Motivation/Lemma
  - ▶ Application of the lemma ( $a^n b^n, a^n b^m, n < m$ )
  - ▶ Implications (Nested structures are not regular)
- ▶ Properties of regular languages
  - ▶ Closure properties (union, intersection, ...)
  - ▶ Proof per construction of suitable NFA (union, concatenation, Kleene Star)
  - ▶ Proof per construction of product automaton (intersection)

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

# Goals for Lecture 9

- ▶ Completing the theory of regular languages
  - ▶ Complement
  - ▶ Finite languages
  - ▶ Decision problems: Emptiness, word problem, . . .
  - ▶ Decision problems: Equivalence, finiteness, . . .
  - ▶ Wrap-up

## ▶ The Pumping Lemma

### ▶ Motivation/Lemma

- ▶ For every regular language  $L$  there exists a  $k$  such that any word  $s$  with  $|s| \geq k$  can be split into  $s = uvw$  with  $|uv| \leq k$  and  $v \neq \varepsilon$  and  $uv^h w \in L$  for all  $h \in \mathbb{N}$
- ▶ Use in proofs by contradiction: Assume a language is regular, then derive contradiction

- ▶ Application of the lemma ( $a^n b^n, a^n b^m, n < m$ )
- ▶ Implications (Nested structures are not regular)

## ▶ Properties of regular languages

- ▶ The union of two regular languages is regular
- ▶ The intersection of two regular languages is regular (Solution: Product automaton!)
- ▶ The concatenation of two regular languages is regular
- ▶ The Kleene star of a regular language is regular
- ▶ (The complement of a regular language is regular)



- ▶ Completing the theory of regular languages
  - ▶ Complement
  - ▶ All finite languages are regular
  - ▶ Decidable for regular languages: Emptiness, word problem
  - ▶ Decidable for regular languages: Equivalence, finiteness
  - ▶ Wrap-up

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

# Goals for Lecture 10

- ▶ Review of last lecture
- ▶ Scanning in practice
  - ▶ Scanners in context
  - ▶ Practical regular expressions
  - ▶ Automatic scanner creation with `flex`
  - ▶ `Flex` exercise

- ▶ Completing the theory of regular languages
  - ▶ Complement
  - ▶ All finite languages are regular
  - ▶ Decidable for regular languages: Emptiness, word problem
  - ▶ Decidable for regular languages: Equivalence, finiteness
  - ▶ Use of properties to prove statements (e.g. non-regularity, e.g. reducing emptiness question to equivalency)
  - ▶ Wrap-up

# Summary

- ▶ Scanners in context
- ▶ Practical regular expressions
  - ▶ Basic characters, escapes, ranges, escape with \
  - ▶ Richer operators, z.B. +, {4},?...
- ▶ Flex
  - ▶ Definition section
  - ▶ Rule section
  - ▶ User code section/`yylex()`
- ▶ Exercise

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions



# Goals for Lecture 11

- ▶ Review of last lecture
- ▶ Grammars and languages
  - ▶ Formal grammars
  - ▶ Derivations
  - ▶ Languages
- ▶ The Chomsky-Hierarchy

# Review

- ▶ Scanners in context
- ▶ Practical regular expressions
  - ▶ Basic characters, escapes, ranges, escape with \
  - ▶ Richer operators, z.B. +, {4},?...
- ▶ Flex
  - ▶ Definition section
  - ▶ Rule section
  - ▶ User code section/`yylex()`
- ▶ make
- ▶ Exercise

- ▶ Grammars and languages
  - ▶ Formal grammars
  - ▶ Derivations
  - ▶ Languages
  - ▶ Grammars can describe non-regular languages
- ▶ The Chomsky-Hierarchy
  - ▶ Type 0: Unrestricted grammars
  - ▶ Type 1: Context-sensitive/monotonic grammars
  - ▶ Type 2: Context-free grammars
  - ▶ Type 3: Right-linear/regular grammars

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

# Goals for Lecture 12

- ▶ Review of last lecture
- ▶ Right-linear grammars
  - ▶ Equivalence with finite automata
- ▶ Context-free grammars
  - ▶ Equivalency for grammars
  - ▶ Reducing grammars
  - ▶ Eliminating  $\epsilon$ -Productions
  - ▶ Chomsky-hierarchy revisited

- ▶ Grammars and languages
  - ▶ Formal grammars  $(N, \Sigma, P, S)$
  - ▶ Derivations
  - ▶ Languages  $L(G) = \{w \in \Sigma^* \mid S \Rightarrow^* w\}$
  - ▶ Grammars can describe non-regular languages
- ▶ The Chomsky-Hierarchy
  - ▶ Type 0: Unrestricted grammars
  - ▶ Type 1: Context-sensitive/monotonic grammars (non-shortening rules)
  - ▶ Type 2: Context-free grammars (Only one non-terminal on LHS)
  - ▶ Type 3: Right-linear/regular grammars (limited RHS)

# Summary

- ▶ Right-linear grammars:  $N \rightarrow aB, a \in \Sigma \cup \{\epsilon\}$ 
  - ▶ Convert DFA to right-linear grammar
  - ▶ Convert right-linear grammar to NFA
- ▶ Context-free grammars
  - ▶ Chomsky Normal Form
  - ▶ Towards CNF
    - ▶ Remove non-terminating symbols and corresponding rules
    - ▶ Remove non-reachable symbols and corresponding rules
    - ▶ Inline  $\epsilon$ -rules



# Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

# Goals for Lecture 13

- ▶ Review of last lecture
- ▶ Completing Chomsky Normal Form transformation
  - ▶ Eliminating chain rules
  - ▶ Bring right hand sides into normalform
- ▶ Solving the word problem for context-free grammars
  - ▶ Derivations in CNF
  - ▶ Parsing with Dynamic Programming: Cocke-Younger-Kasami

- ▶ Right-linear grammars:  $N \rightarrow aB, a \in \Sigma \cup \{\epsilon\}$ 
  - ▶ Convert DFA to right-linear grammar ( $N \sim Q, \delta \sim P$ )
  - ▶ Convert right-linear grammar to NFA
- ▶ Context-free grammars
  - ▶ Chomsky Normal Form
  - ▶ Towards CNF
    - ▶ Remove non-terminating symbols and corresponding rules
    - ▶ Remove non-reachable symbols and corresponding rules
    - ▶ Inline  $\epsilon$ -rules
    - ▶ ...

# Summary

- ▶ Chomsky Normal Form transformation
  - ▶ Inline  $\epsilon$ -rules
  - ▶ Inline chain rules
  - ▶ Reduce grammar
  - ▶ Introduce NTS names for terminals
  - ▶ Break long RHS via introduction of definitions/intermediate rules
- ▶ Solving the word problem for context-free grammars
  - ▶ Derivations in CNF
  - ▶ Cocke-Younger-Kasami
    - ▶ Systematically consider all decompositions of  $w$
    - ▶ Tabulate all NTS for given subwords bottom-up
    - ▶ Example of *dynamic programming*

# Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

# Goals for Lecture 14

- ▶ Review of last lecture
- ▶ Pushdown automata
  - ▶ NFA+unlimited stack
- ▶ Equivalence of context-free grammars and PDAs
  - ▶ From grammar to PDA
  - ▶ From PDA to grammar



- ▶ Chomsky Normal Form transformation
  - ▶ Inline  $\varepsilon$ -rules
  - ▶ Inline chain rules
  - ▶ Reduce grammar
  - ▶ Introduce NTS names for terminals
  - ▶ Break long RHS via introduction of definitions/intermediate rules
- ▶ Solving the word problem for context-free grammars
  - ▶ Derivations in CNF
  - ▶ Cocke-Younger-Kasami
    - ▶ Systematically consider all decompositions of  $w$
    - ▶ Tabulate all NTS for given subwords bottom-up
    - ▶ Complexity:  $O(n^3)$  with  $n = |w|$

# Summary

- ▶ Pushdown automata
  - ▶ Transitions must read/can write unlimited stack
  - ▶ Transition can read characters from word (left-to-right only)
  - ▶ Acceptance: Empty stack, empty word
- ▶ Equivalence of context-free grammars and PDAs
  - ▶ From grammar to PDA
    - ▶ Simulate grammar rules on the stack
    - ▶ Highly non-deterministic
  - ▶ From PDA to grammar
    - ▶ Non-terminals represent state changes with removal of a symbols from stack
    - ▶ Complex encoding of transition relation into grammar rules

# Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Bonus Exercises

Selected Solutions

# Goals for Lecture 15

- ▶ Review of last lecture
- ▶ Limits of context-free languages
  - ▶ Pumping Lemma II
- ▶ Closure properties of context-free languages
- ▶ Decision problems for context-free languages

- ▶ Pushdown automata
  - ▶ Transitions must read/can write unlimited stack
  - ▶ Transition can read characters from word (left-to-right only)
  - ▶ Acceptance: Empty stack, empty word
- ▶ Equivalence of context-free grammars and PDAs
  - ▶ From grammar to PDA
    - ▶ Simulate grammar rules on the stack
    - ▶ Highly non-deterministic
  - ▶ From PDA to grammar
    - ▶ Non-terminals represent state changes with removal of a symbols from stack
    - ▶ Complex encoding of transition relation into grammar rules

# Summary

- ▶ Limits of context-free languages
  - ▶ Pumping Lemma II
- ▶ Closure properties of context-free languages
  - ▶ The class of context-free languages is closed under  $\cup, \cdot, *$
  - ▶ The class of context-free languages is not closed under  $\cap$ , complement
- ▶ Decision problems for context-free languages
  - ▶ Decidable: Word problem, emptiness problem
  - ▶ Undecidable: Equivalence

# Feedback round

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?



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Selected Solutions

# Goals for Lecture 16

- ▶ Review of last lecture
- ▶ Introduction to parsing
- ▶ YACC/Bison
  - ▶ Automatic parser generation
  - ▶ Example: desk calculator
- ▶ LALR(1) shift/reduce parsing
- ▶ Parse trees and abstract syntax trees

- ▶ Limits of context-free languages
  - ▶ Pumping Lemma II
    - ▶ Sufficiently long words require correspondingly long derivations
    - ▶ Sufficiently long derivations will contain at least one duplicate NTS  $A$
    - ▶ The partial derivation from  $A$  to  $\mu Av$  can be repeated
- ▶ Closure properties of context-free languages
  - ▶ The class of context-free languages is closed under  $\cup, \cdot, *$
  - ▶ The class of context-free languages is not closed under  $\cap$ , complement
- ▶ Decision problems for context-free languages
  - ▶ Decidable: Word problem, emptiness problem
  - ▶ Undecidable: Equivalence

# Summary

- ▶ Introduction to parsing
  - ▶ Recognize words (programs) in  $L(G)$
  - ▶ Understand structure of words
- ▶ YACC/Bison
  - ▶ Automatic parser generation
    - ▶ Core: Syntax rules with actions (C code)
    - ▶ Whenever input can be *reduced*, action is executed
    - ▶ Often: Manipulation of *semantic values*
  - ▶ Example: desk calculator
- ▶ LALR(1) shift/reduce parsing
  - ▶ **Shift** input to stack
  - ▶ **Reduce** RHS to LHS
  - ▶ Use lookahead to reduce ambiguity
- ▶ Parse trees and abstract syntax trees
  - ▶ Parse trees: Represents not just word but derivation
  - ▶ AST: “parse tree without syntactic details

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

# Outline

Introduction

Regular Languages and Finite Automata

Scanners and Flex

Formal Grammars and Context-Free Languages

Parsers and Bison

Turing Machines and Languages of Type 1 and 0

## Lecture-specific material

Lecture 1

Lecture 2

Lecture 3

Lecture 4

Lecture 5

Lecture 6

Lecture 7

Lecture 8

Lecture 9

Lecture 10

Lecture 11

Lecture 12

Lecture 13

Lecture 14

Lecture 15

Lecture 16

**Lecture 17**

Lecture 18

Bonus Exercises

Selected Solutions

# Goals for Lecture 17

- ▶ Review of last lecture
- ▶ Beyond context-free languages: Turing Machines
  - ▶ Idea
  - ▶ Definition
  - ▶ Examples
  - ▶ Variants
- ▶ Context-sensitive languages
  - ▶ Linear bounded Turing Machine (LBA)
  - ▶ Closure properties
  - ▶ Decision problems
- ▶ Type-0 languages
  - ▶ The Universal Turing Machine
  - ▶ The Halting Problem
- ▶ Thesis of Church-Turing
  - ▶ ... and practical applications

# Review

- ▶ Introduction to parsing
  - ▶ Recognize words (programs) in  $L(G)$
  - ▶ Understand structure of words
- ▶ YACC/Bison
  - ▶ Automatic parser generation
    - ▶ Core: Syntax rules with actions (C code)
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- ▶ Parse trees and abstract syntax trees
  - ▶ Parse trees: Represents not just word but derivation
  - ▶ AST: “parse tree without syntactic details”



# Summary

- ▶ Turing Machines
  - ▶ Unbounded tape memory
  - ▶ Finite automaton controls head movement, writing based on current state and input at current tape location
  - ▶ Configurations (general, stopping and accepting)
  - ▶ TM for  $a^n b^n c^n$
- ▶ Context-sensitive languages
  - ▶ Linear bounded Turing Machine (LBA)
  - ▶ Closure properties: Closed under  $\cup, \cdot, *, \cap, \bar{\quad}$
  - ▶ Decision problems: Word yes, emptiness, equivalence no
- ▶ Type-0 languages
  - ▶ The Universal Turing Machine
  - ▶ The Halting Problem
- ▶ Thesis of Church-Turing: All “strong” computing models are equivalent
  - ▶ ... and so may be humans (!)

- ▶ What was the best part of today's lecture?
- ▶ What part of today's lecture has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Lecture 12

Lecture 13

Lecture 14

Lecture 15

Lecture 16

Lecture 17

**Lecture 18**

Bonus Exercises

Selected Solutions

## Test Exam

# Summary

- ▶ Review of last lecture
- ▶ Test exam
- ▶ Solutions

# Final feedback round

- ▶ What was the best part of the **course**?
- ▶ What part of the course that has the most potential for improvement?
  - ▶ Optional: how would you improve it?

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Lecture-specific material

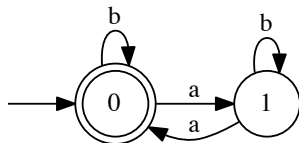
**Bonus Exercises**

Selected Solutions

# Bonus Exercises

Assume  $\Sigma = \{a, b\}$ . Consider the automaton  $A$ .

- ▶ Give a formal description of  $A$ .
- ▶ Which language  $L(A)$  is accepted by  $A$ ? Give a formal description.
- ▶ Where does  $L(A)$  reside in the Chomsky-hierarchy?
- ▶ Manually find a regular expression  $R$  with  $L(R) = L(A)$ .
- ▶ Generate a system of equations for  $A$  and generate a regular expression  $R_S$  for  $L(A)$  by solving this. Compare your result the the previous one.
- ▶ Convert  $A$  into a regular grammar  $G$  with  $L(A) = L(G)$ .



- ▶ Can  $L(A)$  be pumped? If yes, provide an example of a pumpable word.
- ▶ Generalize  $A$  to recognize  $L_3 = \{w \in \Sigma^* \mid |w|_a \text{ modulo } 3 = 0\}$
- ▶ Form the product automaton  $P$  to find  $L(A) \cap L_3$ .
- ▶ Minimize  $P$ .
- ▶ Systematically construct a NFA  $D_A$  for  $R_S$ .
- ▶ Convert  $D_R$  to a DFA.



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**Selected Solutions**

# Equivalence of regular expressions

## Solution to Exercise: Algebra on regular expressions (1)

► Claim:  $r^* \doteq \varepsilon + r^*$

$$\varepsilon + r^* \doteq \varepsilon + \varepsilon + r^*r \quad (13)$$

Proof:  $\doteq \varepsilon + r^*r \quad (9)$

$$\doteq r^* \quad (13)$$

# Simplification of regular expressions

## Solution to Exercise: Algebra on regular expressions (2)

$$\begin{aligned}r &= 0(\varepsilon + 0 + 1)^* + (\varepsilon + 1)(1 + 0)^* + \varepsilon \\ &\stackrel{14,1}{\doteq} 0(0 + 1)^* + (\varepsilon + 1)(0 + 1)^* + \varepsilon \\ &\stackrel{7}{\doteq} 0(0 + 1)^* + \varepsilon(0 + 1)^* + 1(0 + 1)^* + \varepsilon \\ &\stackrel{5}{\doteq} 0(0 + 1)^* + (0 + 1)^* + 1(0 + 1)^* + \varepsilon \\ &\stackrel{1,7}{\doteq} \varepsilon + (0 + 1)(0 + 1)^* + (0 + 1)^* \\ &\stackrel{16}{\doteq} \varepsilon + (0 + 1)^*(0 + 1) + (0 + 1)^* \\ &\stackrel{13}{\doteq} (0 + 1)^* + (0 + 1)^* \\ &\stackrel{9}{\doteq} (0 + 1)^*.\end{aligned}$$

# Application of Arto's lemma

## Solution to Exercise: Algebra on regular expressions (3)

▶ Show that  $u = 10(10)^* \doteq 1(01)^*0$

▶ Idea:  $u$  is of the form  $ts^*$  with:

▶  $t = 10$

▶  $s = 10$

▶ This suggest Arto's Lemma. To apply the lemma, we must show that  $r = 1(01)^*0 \doteq rs + t$

$$\begin{aligned}rs + t &= 1(01)^*010 + 10 \\ &\doteq 1((01)^*010 + 0) \quad \text{(factor out 1)}\end{aligned}$$

▶ So:  $\doteq 1((01)^*01 + \varepsilon)0 \quad \text{(factor out 0)}$

$$\doteq 1(01)^*0 \quad (1),(13)$$

$$= r$$

▶ Since  $L(s) = \{10\}$  (and hence  $\varepsilon \notin L(s)$ ), we can apply Arto and rewrite  $r \doteq ts^* \doteq 10(10)^*$ .

# Transformation into DFA (1)

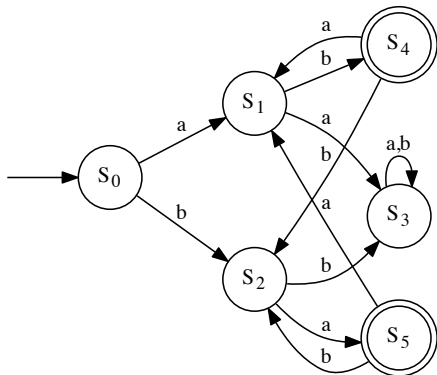
- ▶ Incremental computation of  $\hat{Q}$  and  $\hat{\delta}$ :
  - ▶ Initial state  $S_0 = ec(q_0) = \{q_0, q_1, q_2\}$
  - ▶  $\hat{\delta}(S_0, a) = \delta^*(q_0, a) \cup \delta^*(q_1, a) \cup \delta^*(q_2, a) = \{\} \cup \{\} \cup \{q_4\} = \{q_4\} = S_1$
  - ▶  $\hat{\delta}(S_0, b) = \{q_3\} = S_2$
  - ▶  $\hat{\delta}(S_1, a) = \{\} = S_3$
  - ▶  $\hat{\delta}(S_1, b) = ec(q_6) = \{q_6, q_7, q_0, q_1, q_2\} = S_4$
  - ▶  $\hat{\delta}(S_2, a) = \{q_5, q_7, q_0, q_1, q_2\} = S_5$
  - ▶  $\hat{\delta}(S_2, b) = \{\} = S_3$
  - ▶  $\hat{\delta}(S_3, a) = \{\} = S_3$
  - ▶  $\hat{\delta}(S_3, b) = \{\} = S_3$
  - ▶  $\hat{\delta}(S_4, a) = \{q_4\} = S_1$
  - ▶  $\hat{\delta}(S_4, b) = \{q_3\} = S_2$
  - ▶  $\hat{\delta}(S_5, a) = \{q_4\} = S_1$
  - ▶  $\hat{\delta}(S_5, b) = \{q_3\} = S_2$
- ▶  $\hat{F} = \{S_4, S_5\}$  (since  $q_7 \in S_4, q_7 \in S_5$ )

# Transformation into DFA (2)

- ▶  $det(\mathcal{A}) = (\hat{Q}, \Sigma, \hat{\delta}, S_0, \hat{F})$ 
  - ▶  $\hat{Q} = \{S_0, S_1, S_2, S_3, S_4, S_5\}$
  - ▶  $\hat{F} = \{S_4, S_5\}$
  - ▶  $\hat{\delta}$  given by the table below

$\hat{\delta}$	$a$	$b$
$\rightarrow S_0$	$S_1$	$S_2$
$S_1$	$S_3$	$S_4$
$S_2$	$S_5$	$S_3$
$S_3$	$S_3$	$S_3$
$*S_4$	$S_1$	$S_2$
$*S_5$	$S_1$	$S_2$

- ▶ Regexp:  
 $L(\mathcal{A}) = L((ab + ba)(ab + ba)^*)$

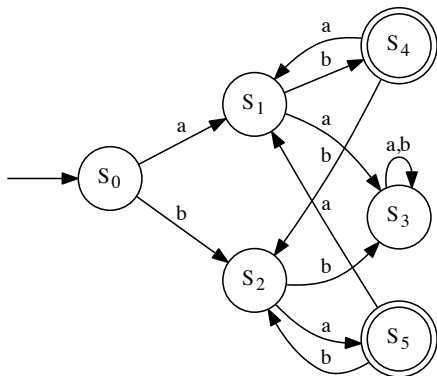


## Transformation into DFA (2)

- ▶  $det(\mathcal{A}) = (\hat{Q}, \Sigma, \hat{\delta}, S_0, \hat{F})$ 
  - ▶  $\hat{Q} = \{S_0, S_1, S_2, S_3, S_4, S_5\}$
  - ▶  $\hat{F} = \{S_4, S_5\}$
  - ▶  $\hat{\delta}$  given by the table below

$\hat{\delta}$	$a$	$b$
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$S_1$	$S_3$	$S_4$
$S_2$	$S_5$	$S_3$
$S_3$	$S_3$	$S_3$
$*S_4$	$S_1$	$S_2$
$*S_5$	$S_1$	$S_2$

- ▶ Regexp:  
 $L(\mathcal{A}) = L((ab + ba)(ab + ba)^*)$

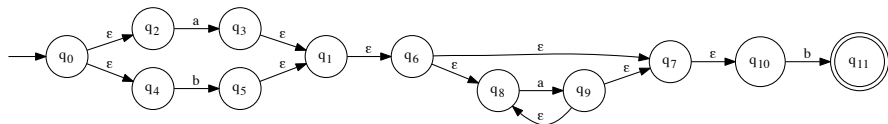


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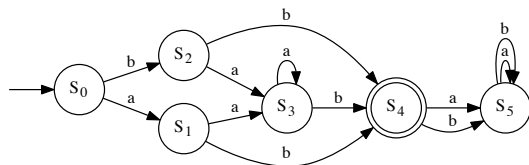
# Transformation of RE into NFA

Systematically construct an NFA accepting the same language as the regular expression  $(a + b)a^*b$ .

Solution:



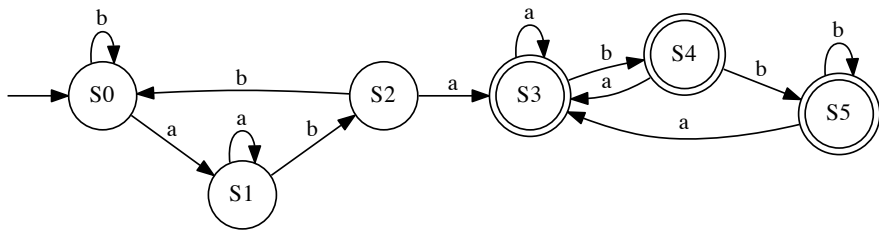
Corresponding DFA:



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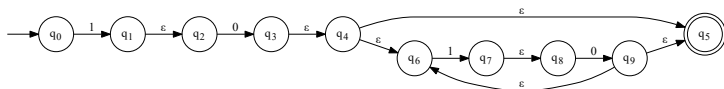
# Solution: NFA to DFA “aba”



# Show $10(10)^* \doteq 1(01)^*0$ via minimal DFAs (1)

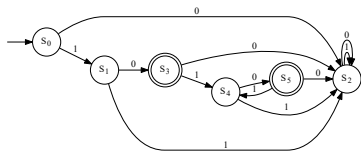
► Step 1: NFA for  $10(10)^*$ :

	epsilon	0	1
-> q0	{}	{}	{q1}
q1	{q2}	{}	{}
q2	{}	{q3}	{}
q3	{q4}	{}	{}
q4	{q5, q6}	{}	{}
* q5	{}	{}	{}
q6	{}	{}	{q7}
q7	{q8}	{}	{}
q8	{}	{q9}	{}
q9	{q5, q6}	{}	{}



# Show $10(10)^* \doteq 1(01)^*0$ via minimal DFAs (2)

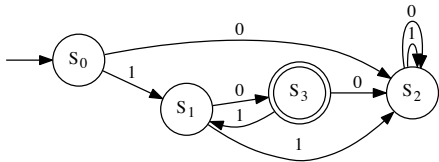
- Step 2: DFA  $\mathcal{A}$  for  $10(10)^*$ :



- Step 3: Minimizing of  $\mathcal{A}$

	$S_0$	$S_1$	$S_2$	$S_3$	$S_4$	$S_5$
$S_0$	0	X	X	X	X	X
$S_1$	X	0	X	X	0	X
$S_2$	X	X	0	X	X	X
$S_3$	X	X	X	0	X	0
$S_4$	X	0	X	X	0	X
$S_5$	X	X	X	0	X	0

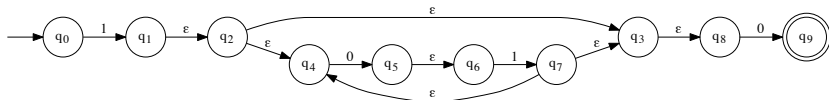
Result:  $(S_1, S_4)$  and  $(S_3, S_5)$  can be merged



# Show $10(10)^* \doteq 1(01)^*0$ via minimal DFAs (3)

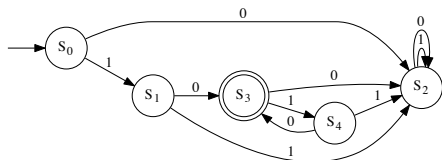
► Step 4: NFA zu  $1(01)^*0$ :

	epsilon	0	1
-> q0	{}	{}	{q1}
q1	{q2}	{}	{}
q2	{q3, q4}	{}	{}
q3	{q8}	{}	{}
q4	{}	{q5}	{}
q5	{q6}	{}	{}
q6	{}	{}	{q7}
q7	{q4, q3}	{}	{}
q8	{}	{}	{q9}
* q9	{}	{}	{}



# Show $10(10)^* \doteq 1(01)^*0$ via minimal DFAs (4)

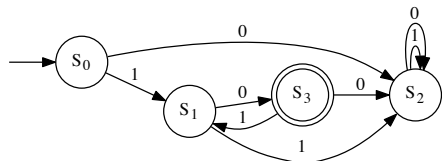
- Step 5: DFA  $\mathcal{B}$  for  $1(01)^*0$



- Step 6: Minimization of  $\mathcal{B}$

	$s_0$	$s_1$	$s_2$	$s_3$	$s_4$
$s_0$	o	x	x	x	x
$s_1$	x	o	x	x	o
$s_2$	x	x	o	x	x
$s_3$	x	x	x	o	x
$s_4$	x	o	x	x	o

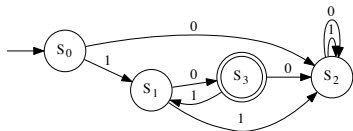
Result:  $(s_1, s_4)$  can be merged



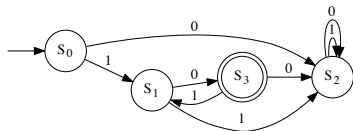
# Show $10(10)^* \doteq 1(01)^*0$ via minimal DFAs (5)

- ▶ Step 7: Comparison of  $\mathcal{A}^-$  and  $\mathcal{B}^-$

$\mathcal{A}^-$



$\mathcal{B}^-$



- ▶ Result: The two automata are identical, hence the two original regular expressions describe the same languages.

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# Pumping lemma

Solution to  $a^n b^m$  with  $n < m$

- ▶ Proposition:  $L = \{a^n b^m \mid n < m\}$  is not regular.
- ▶ Proof by contradiction. We assume  $L$  is regular
- ▶ Then:  $\exists k \in \mathbb{N}$  with:
  - ▶  $\forall s \in L$  with  $|s| \geq k : \exists u, v, w \in \Sigma^*$  such that
    - ▶  $s = uvw$
    - ▶  $|uv| \leq k$
    - ▶  $v \neq \varepsilon$
    - ▶  $uv^h w \in L$  for all  $h \in \mathbb{N}$
- ▶ We consider the word  $s = a^k b^{k+1} \in L$ 
  - ▶ Since  $|uv| \leq k$ :  $u = a^i, v = a^j, w = a^l b^{k+1}$  and  $j > 0, i + j + l = k$
  - ▶ Now consider  $s' = uv^2 w$ . According to the pumping lemma,  $s' \in L$ . But  $s' = a^i a^j a^j a^l b^{k+1} = a^{i+j+l+j} b^{k+1} = a^{k+j} b^{k+1}$ . Since  $j \in \mathbb{N}, j > 0$ :  $k + j \not< k + 1$ . Hence  $s' \notin L$ . This is a contradiction. Hence the assumption is wrong, and the original proposition is true. q.e.d.

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## Solution: Pumping lemma (Prime numbers)

- ▶ Proposition:  $L = \{a^p \mid p \in \mathbb{P}\}$  is not regular (where  $\mathbb{P}$  is the set of all prime numbers)
- ▶ Proof: By contradiction, using the pumping lemma.
  - ▶ Assumption:  $L$  is regular. Then there exist a  $k$  such that all words in  $L$  with at least length  $k$  can be pumped.
- ▶ Consider the word  $s = a^p$ , where  $p \in \mathbb{P}, p \geq k$ 
  - ▶ Then there are  $u, v, w \in \Sigma^*$  with  $uvw = s, |uv| \leq k, v \neq \varepsilon$ , and  $uv^h w \in L$  for all  $h \in \mathbb{N}$ .
  - ▶ We can write  $u = a^i, v = a^j, w = a^l$  with  $i + j + l = p$
  - ▶ So  $s = a^i a^j a^l$  and  $a^i a^{j \cdot h} a^l \in L$  for all  $h \in \mathbb{N}$ .
  - ▶ Consider  $h = p + 1$ . Then  $a^i a^{j \cdot (p+1)} a^l \in L$
  - ▶  $a^i a^{j \cdot (p+1)} a^l = a^i a^{jp+j} a^l = a^i a^{jp} a^j a^l = a^i a^j a^l a^{jp} = a^p a^{jp} = a^{(j+1)p}$
  - ▶ But  $(j+1)p \notin \mathbb{P}$ , since  $j+1 > 1$  and  $p > 1$ , and  $(j+1)p$  thus has (at least) two non-trivial divisors.
  - ▶ Thus  $a^{(j+1)p} \notin L$ . This violates the pumping lemma and hence contradicts the assumption. Thus the assumption is wrong and the proposition holds. *q.e.d.*

## Solution: Pumping lemma (Prime numbers)

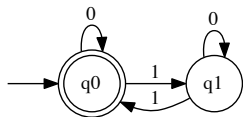
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- ▶ Proof: By contradiction, using the pumping lemma.
  - ▶ Assumption:  $L$  is regular. Then there exist a  $k$  such that all words in  $L$  with at least length  $k$  can be pumped.
- ▶ Consider the word  $s = a^p$ , where  $p \in \mathbb{P}, p \geq k$ 
  - ▶ Then there are  $u, v, w \in \Sigma^*$  with  $uvw = s, |uv| \leq k, v \neq \varepsilon$ , and  $uv^h w \in L$  for all  $h \in \mathbb{N}$ .
  - ▶ We can write  $u = a^i, v = a^j, w = a^l$  with  $i + j + l = p$
  - ▶ So  $s = a^i a^j a^l$  and  $a^i a^{j \cdot h} a^l \in L$  for all  $h \in \mathbb{N}$ .
  - ▶ Consider  $h = p + 1$ . Then  $a^i a^{j \cdot (p+1)} a^l \in L$
  - ▶  $a^i a^{j \cdot (p+1)} a^l = a^i a^{jp+j} a^l = a^i a^{jp} a^j a^l = a^i a^j a^l a^{jp} = a^p a^{jp} = a^{(j+1)p}$
  - ▶ But  $(j+1)p \notin \mathbb{P}$ , since  $j+1 > 1$  and  $p > 1$ , and  $(j+1)p$  thus has (at least) two non-trivial divisors.
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*q.e.d.*

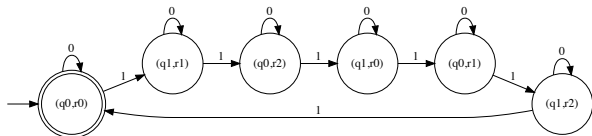
# Solution: Product automaton for $L_1 \cap L_2$

Solution to Exercise: Product automaton

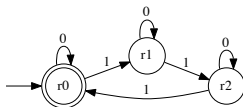
$A_1$ :



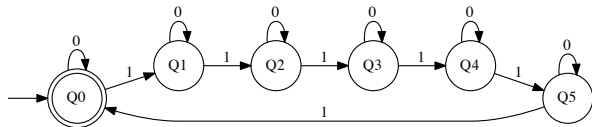
$A_1 \times A_2$  for  $L_1 \cap L_2$ :



$A_2$ :



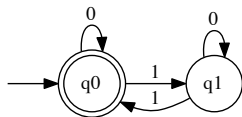
Äquivalent Automaten für  $L_1 \cap L_2$ :



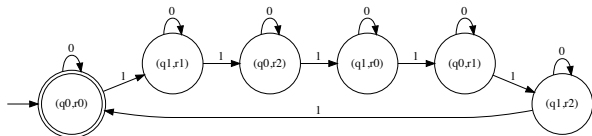
# Solution: Product automaton for $L_1 \cap L_2$

Solution to Exercise: Product automaton

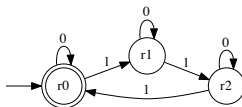
$A_1$ :



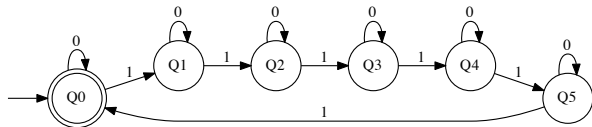
$A_1 \times A_2$  for  $L_1 \cap L_2$ :



$A_2$ :



Äquivalent Automaten für  $L_1 \cap L_2$ :



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# Solution: Transformation to Chomsky Normal Form (1)

Compute the Chomsky normal form of the following grammar:

$$G = (N, \Sigma, P, S)$$

▶  $N = \{S, A, B, C, D, E\}$

▶  $\Sigma = \{a, b\}$

$$S \rightarrow AB|SB|BDE$$

$$C \rightarrow SB$$

▶  $P :$   $A \rightarrow Aa$

$$D \rightarrow E$$

$$B \rightarrow bB|BaB|ab$$

$$E \rightarrow \varepsilon$$

Step 1:  $\varepsilon$ -Elimination

▶ Nullable NTS:  $N = \{E, D\}$

$$S \rightarrow BD \quad (\text{from } S \rightarrow BDE, \beta_1 = BD, \beta_2 = \varepsilon)$$

▶ New rules:  $S \rightarrow BE \quad (\text{from } S \rightarrow BDE, \beta_1 = B, \beta_2 = E)$

$$S \rightarrow B \quad (\text{from } S \rightarrow BD \text{ or } S \rightarrow BE, \beta_1 = B, \beta_2 = \varepsilon)$$

$$D \rightarrow \varepsilon \quad (\text{from } D \rightarrow E, \beta_1 = \varepsilon, \beta_2 = \varepsilon)$$

▶ Remove  $E \rightarrow \varepsilon, D \rightarrow \varepsilon$

## Solution: Transformation to Chomsky Normal Form (2)

### Step 2: Elimination of Chain Rules.

- ▶ Current chain rules:  $S \rightarrow B, D \rightarrow E$
- ▶ Eliminate  $S \rightarrow B$ :
  - ▶  $N(S) = \{B\}$
  - ▶ New rules:  $S \rightarrow bB, S \rightarrow BaB, S \rightarrow ab$
- ▶ Eliminate  $D \rightarrow E$ 
  - ▶  $N(D) = \{E\}$
  - ▶  $E$  has no rule, therefore no new rules!
- ▶ Current state of  $P$ :

$$\begin{aligned} S &\rightarrow AB|SB|BDE|BD|BE|bB|BaB|ab \\ A &\rightarrow Aa \end{aligned}$$
$$\begin{aligned} C &\rightarrow SB \\ B &\rightarrow bB|BaB|ab \end{aligned}$$

# Solution: Transformation to Chomsky Normal Form (3)

## Step 3: Reducing the grammar

- ▶ Terminating symbols:  $T = \{S, B, C\}$  ( $A, D, E$  do not terminate)

- ▶ Remove all rules that contain  $A, E, D$ . Remaining:

$$S \rightarrow SB|bB|BaB|ab \quad C \rightarrow SB$$

$$B \rightarrow bB|BaB|ab$$

- ▶ Reachable symbols:  $R = \{S, B\}$  ( $C$  is not reachable)

- ▶ Remove all rules containing  $C$ . Remaining:

$$S \rightarrow SB|bB|BaB|ab$$

$$B \rightarrow bB|BaB|ab$$

## Solution: Transformation to Chomsky Normal Form (4)

Step 4: Introduce new non-terminals for terminals

- ▶ New rules:  $X_a \rightarrow a, X_b \rightarrow b$ . Result:

$$\begin{array}{ll} S & \rightarrow SB|X_bB|BX_aB|X_aX_b & X_a & \rightarrow a \\ B & \rightarrow X_bB|BX_aB|X_aX_b & X_b & \rightarrow b \end{array}$$

Step 5: Introduce new non-terminals to break up long right hand sides:

- ▶ Problematic RHS:  $BX_aB$  (in two rules)
- ▶ New rule:  $C_1 \rightarrow X_aB$ . Result:

$$\begin{array}{ll} S & \rightarrow SB|X_bB|BC_1|X_aX_b & X_a & \rightarrow a \\ B & \rightarrow X_bB|BC_1|X_aX_b & X_b & \rightarrow b \\ C_1 & \rightarrow X_aB & & \end{array}$$



# Solution: Transformation to Chomsky Normal Form (5)

Final grammar:  $G' = (N', \Sigma, P', S)$  with

▶  $N' = \{S, B, C_1, X_a, X_b\}$

▶  $\Sigma = \{a, b\}$

▶  $P' :$

$S$	$\rightarrow$	$SB X_bB BC_1 X_aX_b$	$X_a$	$\rightarrow$	$a$
$B$	$\rightarrow$	$X_bB BC_1 X_aX_b$	$X_b$	$\rightarrow$	$b$
$C_1$	$\rightarrow$	$X_aB$			

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# Solution: CYK

	1	2	3	4	5	6
1	Y	-	S,B	-	-	S,B
2		X	S,B	-	-	S,B
3			Y	-	-	-
4				X	-	D
5					X	S,B
6						Y
w	b	a	b	a	a	b

$S \rightarrow SB|BD|YB|XY$

$B \rightarrow BD|YB|XY$

$D \rightarrow XB$

$X \rightarrow a$

$Y \rightarrow b$

Therefore  $babaab \in L(G)$

# Solution: CYK

	1	2	3	4
1	X	S, B	-	-
2		Y	-	-
3			Y	-
4				X
w	a	b	b	a

$$S \rightarrow SB|BD|YB|XY$$

$$B \rightarrow BD|YB|XY$$

$$D \rightarrow XB$$

$$X \rightarrow a$$

$$Y \rightarrow b$$

Therefore  $abba \notin L(G)$

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## Solution: PDA to Grammar (1)

$$A = (Q, \Sigma, \Gamma, \Delta, 0, Z)$$

- ▶  $Q = \{0, 1\}$

- ▶  $\Sigma = \{a, b\}$

- ▶  $\Gamma = \{A, Z\}$

$$\Delta :$$

(1)	0	$\epsilon$	Z	$\rightarrow$	$\epsilon$	0
(2)	0	a	Z	$\rightarrow$	AZ	0
(3)	0	a	A	$\rightarrow$	AA	0
(4)	0	b	A	$\rightarrow$	$\epsilon$	1
(5)	1	b	A	$\rightarrow$	$\epsilon$	1
(6)	1	$\epsilon$	Z	$\rightarrow$	$\epsilon$	1

$$G = (N, \Sigma, P, S)$$

- ▶  $N = \{S, [0A0], [0A1], [1A0], [1A1], [0Z0], [0Z1], [1Z0], [1Z1]\}$

- ▶  $\Sigma$  and  $S$  as given

- ▶ Start rules for  $P$ :

- ▶  $S \rightarrow [0Z0]$

- ▶  $S \rightarrow [0Z1]$

- ▶ From transitions:

- ▶ From (1):  $[0Z0] \rightarrow \epsilon$

- ▶ From (4):  $[0A1] \rightarrow b$

- ▶ From (5):  $[1A1] \rightarrow b$

- ▶ From (6):  $[1Z1] \rightarrow \epsilon$

- ▶ (2) and (3): Next page

## Solution: PDA to Grammar (2)

$$\mathcal{A} = (Q, \Sigma, \Gamma, \Delta, 0, Z)$$

▶  $Q = \{0, 1\}$

▶  $\Sigma = \{a, b\}$

▶  $\Gamma = \{A, Z\}$

$$\Delta :$$

(1)	0	$\varepsilon$	Z	$\rightarrow$	$\varepsilon$	0
(2)	0	a	Z	$\rightarrow$	AZ	0
(3)	0	a	A	$\rightarrow$	AA	0
(4)	0	b	A	$\rightarrow$	$\varepsilon$	1
(5)	1	b	A	$\rightarrow$	$\varepsilon$	1
(6)	1	$\varepsilon$	Z	$\rightarrow$	$\varepsilon$	11

Computing of  $P$  continued:

▶ From (2):

▶  $[0Z0] \rightarrow a[0A0][0Z0]$

▶  $[0Z0] \rightarrow a[0A1][1Z0]$

▶  $[0Z1] \rightarrow a[0A0][0Z1]$

▶  $[0Z1] \rightarrow a[0A1][1Z1]$

▶ From (3):

▶  $[0A0] \rightarrow a[0A0][0A0]$

▶  $[0A0] \rightarrow a[0A1][1A0]$

▶  $[0A1] \rightarrow a[0A0][0A1]$

▶  $[0A1] \rightarrow a[0A1][1A1]$

## Solution: PDA to Grammar (3)

Full grammar  $G = \{N, \Sigma, P, S\}$

▶  $N = \{S, [0A0], [0A1], [1A0], [1A1], [0Z0], [0Z1], [1Z0], [1Z1]\}$

▶  $\Sigma = \{a, b\}$

▶  $P$ :

▶  $S \rightarrow [0Z0]$

▶  $S \rightarrow [0Z1]$

▶  $[0Z0] \rightarrow \varepsilon$

▶  $[0A1] \rightarrow b$

▶  $[1A1] \rightarrow b$

▶  $[1Z1] \rightarrow \varepsilon$

▶  $[0Z0] \rightarrow a[0A0][0Z0]$

▶  $[0Z0] \rightarrow a[0A1][1Z0]$

▶  $[0Z1] \rightarrow a[0A0][0Z1]$

▶  $[0Z1] \rightarrow a[0A1][1Z1]$

▶  $[0A0] \rightarrow a[0A0][0A0]$

▶  $[0A0] \rightarrow a[0A1][1A0]$

▶  $[0A1] \rightarrow a[0A0][0A1]$

▶  $[0A1] \rightarrow a[0A1][1A1]$

▶ **Terminating:**  $T = \{[0Z0], [0A1], [1A1], [1Z1], S, [0Z1]\}$

▶ **Remaining rules:**

**1**  $S \rightarrow [0Z0]$

**2**  $S \rightarrow [0Z1]$

**3**  $[0Z0] \rightarrow \varepsilon$

**4**  $[0A1] \rightarrow b$

**5**  $[1A1] \rightarrow b$

**6**  $[1Z1] \rightarrow \varepsilon$

**7**  $[0Z1] \rightarrow a[0A1][1Z1]$

**8**  $[0A1] \rightarrow a[0A1][1A1]$

▶ **Reachable:**

$R = \{S, [0Z0], [0Z1], [0A1], [1Z1], [1A1]\}$

▶ No change!

## Solution: PDA to Grammar (4)

►  $P$ :

1  $S \rightarrow [0Z0]$

2  $S \rightarrow [0Z1]$

3  $[0Z0] \rightarrow \varepsilon$

4  $[0A1] \rightarrow b$

5  $[1A1] \rightarrow b$

6  $[1Z1] \rightarrow \varepsilon$

7  $[0Z1] \rightarrow a[0A1][1Z1]$

8  $[0A1] \rightarrow a[0A1][1A1]$

► Derivation of  $\varepsilon$ :

►  $S \Rightarrow_1 [0Z0] \Rightarrow_3 \varepsilon$

► Derivation of  $ab$ :

►  $S \Rightarrow_2 [0Z1] \Rightarrow_7 a[0A1][1Z1] \Rightarrow_4$   
 $ab[1Z1] \Rightarrow_6 ab$

► Derivation of  $aabb$ :

►  $S \Rightarrow_2 [0Z1] \Rightarrow_7 a[0A1][1Z1] \Rightarrow_8$   
 $aa[0A1][1A1][1Z1] \Rightarrow_4$   
 $aab[1A1][1Z1] \Rightarrow_5 aabb[1Z1] \Rightarrow_6$   
 $aabb$

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